DESCRIPTIVE GEOMETRY

MILLAR MACLIN
MARKWARDT SHIELS

Charles Ziemer

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Charles W. Ziemer

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GEORGE E. MAYCOCK

DESCRIPTIVE GEOMETRY

A. V. MILLAR
Asst. Dean, Prof. of Drawing
College of Engineering, University of Wisconsin

E. S. MACLIN
Prof. of Industrial Education
West Virginia University

L. J. MARKWARDT
Asst. in Charge, Section of Timber Mechanics
U. S. Forest Products Laboratory

K. G. SHIELS
Asst. Prof. of Drawing
College of Engineering, University of Wisconsin

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A. V. MILLAR
E. S. MACLIN
L. J. MARKWARDT
K. G. SHIELS

PREFACE

This text book on descriptive geometry has been used, in various editions, at the University of Wisconsin for the past sixteen years with results which have been most satisfactory and gratifying. In preparing it, the authors have endeavored to present the subject in a simle, logical manner and in close conformity to commercial drafting practice, as well as to develop in the student the faculty of visualizing magnitudes in space.

One of the principal factors in accomplishing these desiderata is the omission of the ground line. When the views of several points are given without the ground line being shown, the distances of the points from the horizontal or vertical planes of projection are not determined. The front view, however, does show the **relative** heights of the points and the top view shows the **relative** distances of the points from the vertical plane. It is the **relative** distances of points of an object from a plane with which we are concerned, since the distance of the whole object from the plane of projection does not change the orthographic projection of the object on that plane.

By the omission of the ground line, and therefore the traces of a plane, the student's attention is concentrated on the object or magnitude in space, and not on the planes of projection. This teaches the student to visualize the object rather than memorize the views of the object and leads to more original work. The subject is thus made easier because memorizing constructions and centering attention on the views of the object rather than on the object itself are

the greatest obstacles which the student encounters in mastering the subject. Since the ground line is omitted in commercial work, the subject as presented in this text conforms to common practice. The only reference to the ground line will be made for convenience in describing direction in space; the ground line being regarded as at right angles to the line joining the top and front views of any point.

It is worthy of note that the *third* quadrant is used almost exclusively in the drafting offices of this country and at the various universities as well in the mechanical drawing courses which are a pre-requisite to descriptive geometry. It seems logical and advisable therefore to present the subject in the third quadrant as is done in this text. Since the elimination of the ground line dispenses with the necessity of using a particular horizontal or vertical plane, it is possible to represent objects such as cones in their natural positions rather than inverting them to bring their bases into a horizontal plane of projection.

Another salient feature of this text is the extensive use of auxiliary views, that is views other than the top, front, or end views. These are taken from any direction which will aid in construction or give a clearer idea of the form of the object. These views train the visualizing powers of the student and in many instances give the most accurate and concise construction for the solution of the problem.

In beginning the study of descriptive geometry, students usually experience considerable difficulty in assuming, on their own initiative, satisfactory layouts for the given exercises and problems, and much valuable time is consumed in this preliminary work. The system of miniature layouts for black board work and the co-ordinate system of presenting problems for drafting room work are features of this text which go far toward relieving these difficulties and allowing the student to concentrate his attention on the solution of the problem.

The authors have consulted many descriptive geometries in preparing the following text, but are particularly indebted for suggestions and problems to the following: Javary, Pillet, "F. J.", MacCord, Smith, Church, Higbee, Ferris, and Cutter.

HISTORICAL SKETCH

Gaspard Monge (1746-1818) was the first person to arrange the problems of descriptive geometry in logical order and to point out fundamental principles contained in them. Previous to this, geometrical magnitudes were represented by means of drawings, and problems in space were solved by means of constructions in a plane. Even before the time of Monge, a Frenchman by the name of Frezier (1682-1773) did much work along the line of descriptive geometry. It remained for Monge to introduce the line of intersection of the horizontal and vertical planes of projection and to revolve one of these planes into the other about this line as an axis.

Gaspard Monge studied and later taught in the annex to the military college for engineers at Mezieres, France. As a student he found that the operations connected with the constructions of plans of fortifications were made by long arithmetical processes and he at once substituted for this a geometric method of his own devising. At first the officer in charge would not accept the graphical solution to problems but later did accept it when he found it superior to methods previously used. In 1768 Monge was made a

professor in the college with the understanding that the results of his descriptive geometry were to be a military secret confined to officers of certain ranks. He further developed the principles of the subject but because of the rivalry between French military schools he did not publish his methods and results until 1795. In this same year Monge began teaching in the Polytechnic School of Paris and continued in this position until he left France to accompany Napoleon on the Egyptian campaign. While in that country he was made the first president of the Institute of Egypt.

From the technical schools of France, descriptive geometry was introduced into similar schools of Germany in the year 1828. In the United States, descriptive geometry was introduced in 1816 at the U. S. Military Academy in West Point by Claude Crozet, once a pupil at the Polytechnic School in Paris. Crozet wrote the first English work on the subject. In 1826, Prof. Davies of the U. S. Military Academy published a text book on the subject. This was followed a few years later by Church's text which for many years was extensively used throughout the United States.

From the U. S. Military Academy at West Point, the study of descriptive geometry has spread throughout the country until at present it is taught in all engineering colleges.

A more complete history of descriptive geometry has been written by Professor F. G. Higbee of the University of Iowa and published in the January, 1929 Journal of Engineering Education.

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DESCRIPTIVE GEOMETRY

CHAPTER I

PRINCIPLES OF PROJECTION AND GENERAL INSTRUCTIONS

1. Descriptive Geometry treats, **first,** of the exact representation of geometrical forms upon planes and, **second,** of the graphical solution on planes of problems involving these forms in space.

Representing an object. Some objects can be described by a written or oral statement in such a way that a clear idea of the size and form of the object may be obtained. For example, the statements "a cube having an edge 3″ long" or "a sphere of 2″ diameter" give a clear idea of the form and size of the objects. But when the object becomes more complicated and it is necessary to show not only its form and size, but also its position in space or its position with reference to some other object, the description by means of an oral or written statement becomes very difficult. A picture or drawing of the object will usually give the desired information regarding it more quickly and accurately than any other form of expression. For this reason the draftsman makes use of working drawings to convey his ideas to the manufacturer.

The principles of descriptive geometry are used in making such working drawings. These principles are usually explained by reference to such geometrical magnitudes as points, lines, and planes. These geometrical magnitudes, however, form the outlines of material bodies; the corners

of a solid are points, the edges lines, and the faces planes or at least surfaces. The principles, therefore, which are explained by means of points, lines, and planes are used in making working drawings of solids. Sheet metal work, bridge and machine design, problems in geology and mining are practical applications of descriptive geometry principles.

The study of descriptive geometry should enable a person to form a mental picture of an object of three dimensions from a drawing made on a plane sheet. This process is called visualizing the object. Visualizing not only enables a person to read a completed drawing, but it also helps the designer to form a mental picture of one part of an object and its position with respect to other parts even before a drawing of the object is made.

There are different methods of representing an object by means of a drawing, but all of the methods use the principle of projection as explained in the following article.

2. **Projection.** If from a point S, Fig. 1, straight lines are drawn through a series of points A, B, C, ..., the points *a*, *b*, *c*, ... in which these lines pierce the plane T, are the **projections** of the points A, B, C. ..... on this plane.

S is the point at which the eye is supposed to be located, and is called the **point of sight.**

A, B, C, ... are points such as corners of an object in space. Sa, Sb, Sc ..... are **lines of sight** or **projecting lines** of the points A, B, C, ...

T is the **picture plane** or the **Plane of projection.**

Leaving color out of consideration, the projections *a*, *b*, *c*, ... present the same appearance to the eye, situated at the point of sight, as the points A, B, C ..... in space.

In **perspective**, Fig. 1, the point of sight is at a **finite distance** from the plane of projection. The projecting lines diverge. Such a view is a true picture of the object, similar to the view obtained with a camera.

In **oblique** and **orthographic** projection, Figs. 2 and 3, the point of sight is at an **infinite distance** from the plane of projection. The projecting lines are **parallel.**

The projection is **oblique** when the projecting lines are parallel to each other and oblique to the plane of projection. In perspective and oblique projections, the picture from one position is considered sufficient to represent the object.

The projection is **orthographic** when the projecting lines are perpendicular to the plane of projection. In this type of projection, one view is not sufficient to represent an object of three dimensions.

Since **orthographic projection** is more commonly used than any other in engineering work, it will be used in the solution of problems given in this text to illustrate the principles of descriptive geometry.

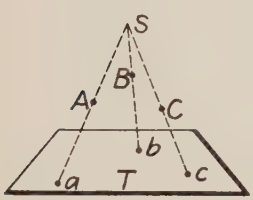


FIG. 1.—*Perspective.*

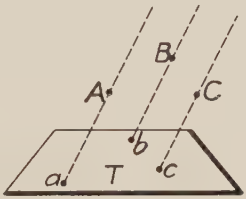


FIG. 2.—*Oblique projection.*

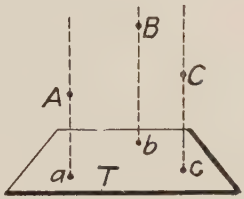


FIG. 3.—*Orthographic projection.*

3. **Views or projections of points.** A point in space is not completely determined by its orthographic projection on one plane, for the distance of the point from the plane is not shown by its projection on that plane. All points A, B, C, Fig. 4, which lie in a vertical straight line have the same projection on a horizontal plane. Any point, P, on a plane T may be considered to be the orthographic projection of all points in space which lie in a perpendicular to plane T at P.

There are two methods of representing definitely a point in space. One method is to give its projection on a plane and also its distance from that plane. As for example in Fig. 4, if *c* is given on the plane T and the statement that the point is 1″ above the plane, then the point C is definitely located in space. This method is used in making contour maps; a contour line being a line joining the projections of all points of an object which are a given distance above or below a base plane. The other method requires the use of two (or more) different planes of projection which are at right angles to each other. When two projections are used, one is usually considered the principal projection. The other is supplementary, showing the distance of the point from the plane upon which the principal projection is made. The method in which two projections of an object are used is the more common.

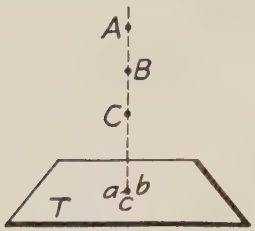


FIG. 4.—*a, b, c, projections of points A, B, C, on plane T.*

4. Views or projections of an object. The **top view** of an object is the view obtained by looking directly down from the top. The point of sight is imagined to be an infinite distance directly above the object. The projecting lines for different points of the object are all vertical and the plane of projection upon which this view or drawing is made is horizontal. This plane is called the **horizontal plane of projection or H.** The top view is also called the **horizontal projection** of the object and in architectural work it is called the **plan.** In Fig. 5, *ab* is the top view or plan of the object.

The **front view** of an object is the view obtained by looking directly from the front. The projecting lines for this view are horizontal and are therefore at right angles to the projecting lines used in obtaining the top view. The plane

upon which the front view is made is in a vertical position, and is called the **vertical plane of projection or V.** The front view is also called the **vertical projection** of the object,

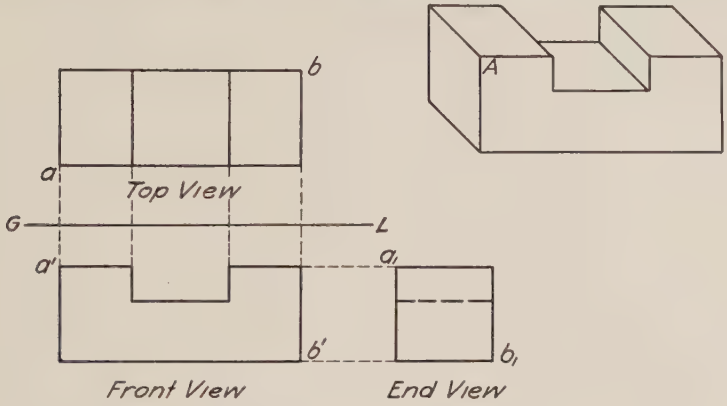


FIG. 5.—*Top, front, and right end views of an object.*

and in architectural work it is called the **front elevation.** In Fig. 5, *a' b'* is the front view or front elevation of the object.

The horizontal and vertical planes of projection are at right angles to each other, Fig. 6. Their line of intersection is called the **ground line or G. L.** In most drawings, the object represented is considered to be below H and back of V. An object in this position is said to be in the **third quadrant or third angle.** In this case the planes of projection are between the points of sight and the object. In some drawings the object is considered to be above H and in front of V. In such a position the object is said to be in the **first quadrant** or **first angle.** Here the object is between the points of sight and the planes of projection. If the object is in the first quadrant, the front view is made above the top view on the drawing, while in third quadrant drawing the front view is below the top view on the drawing, as shown in Fig. 5.

The **end or side view** of an object is the view obtained by looking directly from the left or right. The projecting lines

for this view are parallel to the ground line . The plane upon which this view is made is perpendicular to the ground line and is called an **end** or **profile plane or P** Fig. 6. The end view is also called the **profile projection** and in architectural work the **end or side elevation.** In Fig. 5, a_1 b_1 is the right end view of the object.

5. Relative position of principal views. In Fig. 6, the H, V, and P planes are shown in their proper relative positions. H and V are at right angles to each other, and P is at right angles to both of them, and therefore, at right angles to their line of intersection G.L. If the object is assumed to be in the third quadrant, that is, below H and back of V, the top view of the object is made above the front view on the drawing, as shown in Fig. 5. This is a convenient position in which to make the views since the longer

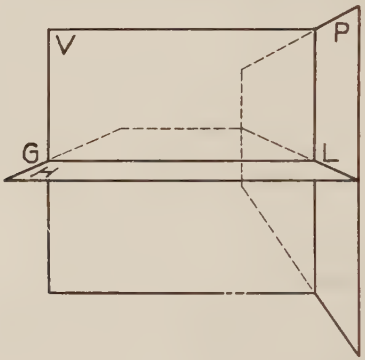


FIG. 6.—*Principal planes of projection.*

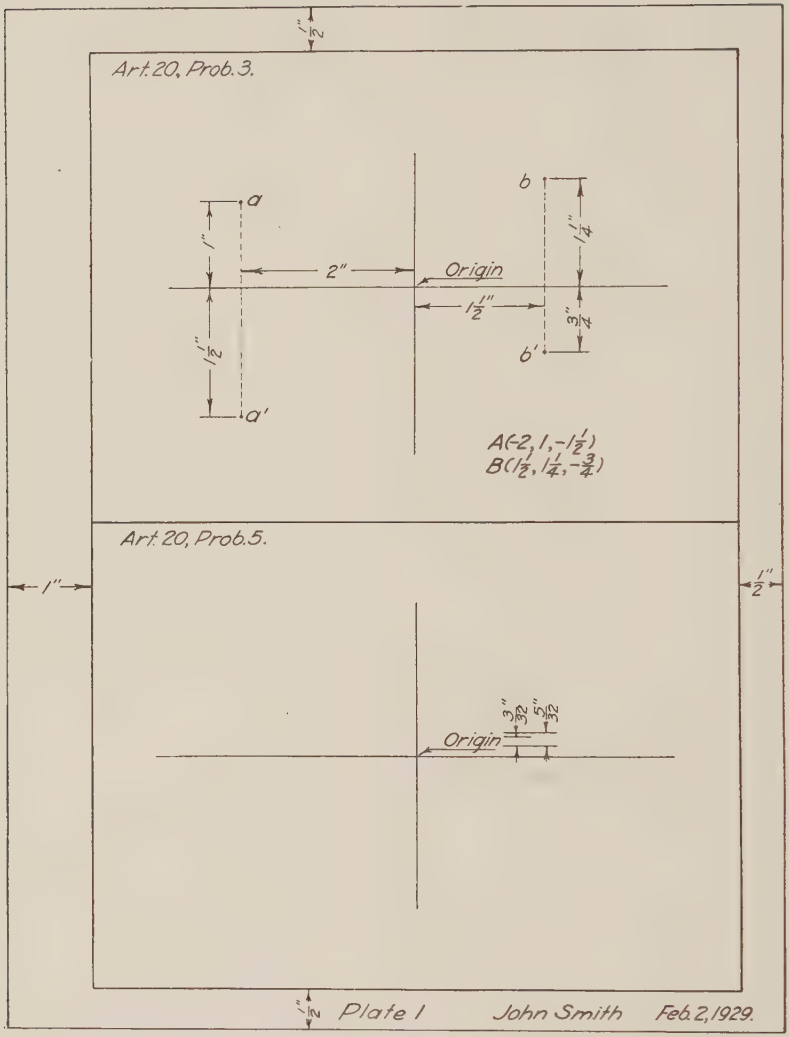
edges of the object are represented by lines drawn parallel to G.L. by means of the T-square and the shorter edges of the object represented by lines drawn at right angles to G.L. by means of a triangle set against the blade of the T-square. The top view a and the front view a' of one corner A of the object should lie in a perpendicular to the ground line. This is true with the top and front views of each corner of the object.

The right end view of the object is usually placed to the right and the left end view to the left of the front view. These are convenient positions since height of the object as shown in the front view can be projected to the right and left end views by means of the T-square. The end views are sometimes placed to the right and left of the top view. In this position, the T-square can be used to project width of the object from the top to the end views.

6. Letter notation. In descriptive geometry it is customary when referring to an object to designate points in space, such as corners, by capital letters, as A, B, C, etc. The top views or horizontal projections of these points are designated by lower case letters, as a, b, c, etc., while the front views are designated by lower case letters primed, as a', b', c', etc. If views other than the top and front of the object are shown, their corners are designated by lower case letters with subscripts, as a_1, b_1, c_1, etc., a_2, b_2, c_2, etc. In Fig. 5, the right end view is lettered a_1, b_1, c_1, etc. If the left end view had been shown also, it would have been lettered a_2, b_2, c_2, etc. Had the left end view been drawn without the right end view, then it would have been lettered a_1, b_1, c_1, etc. The top view of a point P in a rotated position is lettered p_r, the front view p_r', the next views p_{r1}, p_{r2}, etc.

The lines which join two views of the same point are shown in the cuts of this book as fine dashed lines, but it is suggested that on pencil drawings these be made light full lines.

7. Coordinate system. The problems for which coordinates are given in this text can be conveniently solved in a 5½″ x 7½″ rectangle unless statement to the contrary is made. Two problems can be solved on a 9″ x 12″ sheet laid out as shown on page 8. The base line should be drawn through the center and parallel to the longer side of the rectangle. Distances are given in inches and should be used full scale unless otherwise stated. The origin or



Suggested lay out for a 9" x 12" sheet—two problems on a sheet. Co-ordinate system illustrated—height of letters suggested.

starting point for measurement, see page 8, is the center of the base line. The top and front views of any point are located by three coordinates. The first measurement is made from the origin in the direction of the base line, to the right if positive and to the left if negative. At this point located on the base line, draw a light full pencil line perpendicular to the base line and extending both above and below the base line. **The second coordinate** locates the **top view** of the point, above the base line when positive, below when negative. **The third coordinate** locates the **front view** of the point, above the base line when positive, below when negative. The second and third measurements should be made from the point on the base line at which the first measurement stops. For example, the point A (2, 1, —½) has its top view *a* two inches to the right of the origin and one inch above the base line and its front view *a'* two inches to the right of the origin and one-half inch below the base line.

For blackboard work, multiply each coordinate by five. The cuts, accompanying lists of problems, are numbered to correspond with the problems to which they belong. It is suggested that they be used as layouts for blackboard work in place of using the coordinates of the stated problem. The cuts may also be used in a general way to check layouts made on the drawing.

8. **Analysis and construction.** In the solution of a problem in descriptive geometry there are two parts, the analysis and the construction. The *analysis* of the problem states the general method of solution and refers usually to magnitudes in space and not to their views. The analysis for any problem is the same for all different positions of the magnitudes in space. The *construction* of the problem deals with the making of the drawing and explains how the lines are drawn to secure the required result. The construction changes with different positions of the magnitudes in space. The student should endeavor to keep these two

parts of the problem as distinct as possible. The analysis of the problem should be made first and then the construction. For example, let it be required to draw the top and front views of the locus of all points equidistant from three given points A, B, and C in space. The analysis would state that the locus is a straight line perpendicular to the plane ABC at the center of a circle passing through A, B, and C. The construction would involve finding the top and front views of the center of the circle and then drawing the top and front views of the straight line through this point in such directions that the line itself would be perpendicular to the plane ABC.

9. Principal views of points. In Fig. 7, $a\ b\ c$ is the top view, $a'\ b'\ c'$ the front view, and $a_1\ b_1\ c_1$ the right end view of the triangle A B C. When the ground line, G.L. Fig. 6,

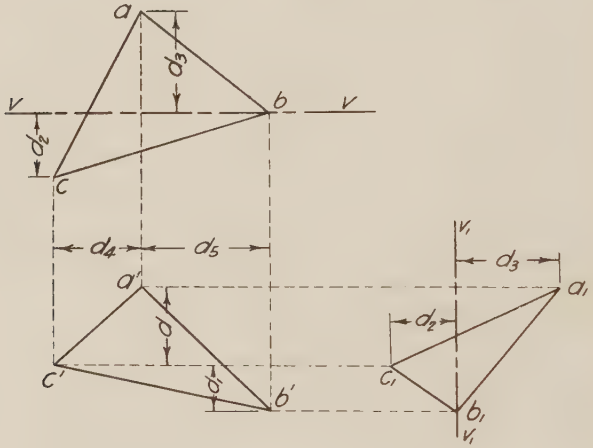


FIG. 7.—*Top, front, and right end views of triangle ABC.*

is omitted from the drawing, the distances of the points A, B, and C, Fig. 7, from the horizontal and vertical planes of projection cannot be determined. However, the size of the triangle A B C, Fig. 7, and its position in space are definitely determined when its top and front views are given even without the ground line being shown. In working

drawings, it is usually the form of the object and its position with respect to other objects with which we are concerned rather than the distance of the object from the planes of projection. We will hereafter use the ground line only as a center line by which to locate the views of an object or as a line in space by means of which we can describe the positions of other lines and planes in space. On the drawing, the ground line is at right angles to the line joining the top and front views of a point, and is usually parallel to the edge of the T-square. In space, it must be pictured as the line of intersection of a horizontal and vertical plane.

The top view, in Fig. 7, shows that A is farther toward the back and C farther toward the front than a vertical plane through point B. The front view shows that A is higher and B lower than point C. From the top and front views, the right end view is determined. The vertical plane V, which appears as line v_1 v_1 in the end view, is placed any convenient distance to the right of the front view. The distances d_2 and d_3, taken from the top view, are used to locate c_1 to the left and a_1 to the right of the vertical plane through b_1 in the right end view. The top and front views of the triangle A B C fix its position in space and from these views other views can be easily constructed.

It is the relative position of points of an object which determines the forms of its views or projections. As an illustration, suppose a three inch cube be placed in a certain position and the top, front, and end views drawn. If each corner of the cube be lowered two inches, the top view of the cube will remain unchanged. The front and end views will remain the same in form, but may be placed lower on the sheet. But if the cube be moved to such a position that one corner is one inch, another two inches, and a third three inches lower than they were originally, all views of the cube will be changed in form. The view of an object, therefore, does not depend upon the distances of its points from the picture plane, but rather upon the relative distances of its points from that plane.

10. Directions for making a drawing. The problems given in this text can be solved conveniently in 5½″ x 7½″, 7½″ x 11″, or 11″ x 16″ rectangles. They are to be solved in 5½″ x 7½″ rectangles unless statement to the contrary is made.

The sheets of drawing paper to be used are of two sizes, 9″ x 12″ and 12″ x 18″. A 9″ x 12″ sheet laid out with margins as shown on page 8, contains one 7½″ x 11″ rectangle or two 5½″ x 7½″ rectangles. A 12″ x 18″ sheet

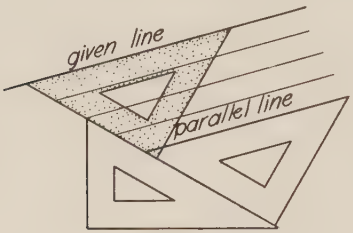


Fig. 8.

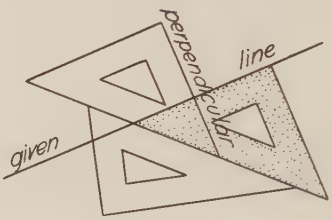


Fig. 9.

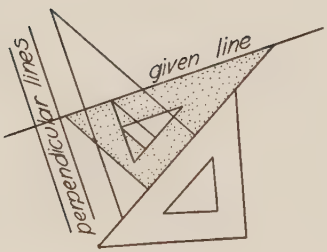


Fig. 10.

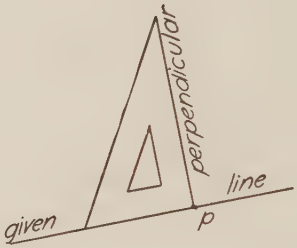


Fig. 11.

The use of triangles in drawing parallel and perpendicular lines. The method illustrated by Fig. 11 cannot be recommended.

with ½″ margins on the longer sides and 1½″ and ½″ margins at the ends, will contain one 11″ x 16″ rectangle, two 8″ x 11″ rectangles, or four 5½″ x 8″ rectangles. The base line is drawn through the center and parallel to the longer side of the 5½″ x 7½″ rectangles and through the

center but parallel to the shorter side of a 7½″ x 11″ or a 11″ x 16″ rectangle. The given points in a problem are located on the drawing by means of coordinates as described in Art. 7 and as shown for points A and B on page 8.

Fig. 8 shows how to use the triangles for drawing parallel lines. Either triangle may be used as the stationary one. Figs. 9 and 10 show the use of the triangles to construct lines perpendicular to a given line. Either method is accurate and easy to use. The method illustrated in Fig. 11 of drawing a perpendicular to a given line at a point P of the line, is not accurate and cannot be recommended.

Either the vertical or inclined style of lettering can be used. There is no objection to using all capitals if the instructor prefers to do so. For drawings the size recommended, the body of the lower case letters should be about 3/32″ high and capitals should be about 5/32″ high. In case all capitals are used, 1/8″ is a good height.

In case the problem solved is given in the text book, it is a good plan, for the sake of future reference, to letter the number of the article and the number of the problem in the rectangle in which the problem is solved, as shown on page 8. If the problem is not given in the text, it is a good plan to letter a complete statement of the problem on the sheet on which the drawing is made.

The number of the plate, the student's name, and the date should be lettered on the drawing. This can be conveniently placed below the border line as shown on page 8.

11. Problems. Solve in orthographic projection the following problems.

1. Draw the top and front views of the points A (—1¼, ½, ?) and B (1¼, 0, —2) when A is ½″ higher than B.

2. Given the top and front views of three points C(—2¼, 1, —2) D (—1¼, ¼, —¾) and E (0, 1½, —1¼). Draw the right end view of the points.

3. Given the front and left end views of three points A (½, ?, —1½) B (1¼, ?, —¾) and C (2¼, ?, —2). The left end views of A, B, and C are ¾″, 2¼″ and 1¾″, respectively, to the left of the origin. Draw the top view of the points.

4. A (—1½, ¾, —1), B (—¾, 1¾, —1), C (½, ¼, —2¼) and E (1, 2¼, —⅛) are the corners of a tetrahedron. Draw the tetrahedron showing hidden edges dashed.

5. Given three points A (—1½, ¾, —1), B (—¾, 1¾, —1) and C (½, ¼, —2¼). Locate the top and front views of a fourth point M, which is 1½″ back of A, 1″ to the right of C, and ½″ lower than B.

12. Picturing magnitudes in space from the drawing. In order to solve problems intelligently, the student should learn to picture to himself points, lines, and objects in their proper positions with reference to the drawing. The best results in picturing or visualizing an object in space can be obtained by considering the positions of the corners and edges of the object with respect to one view of the object, such as the top, front, or end view. The relative distances of the corners of the object from the picture plane on which this view is taken, are obtained from the other views of the object.

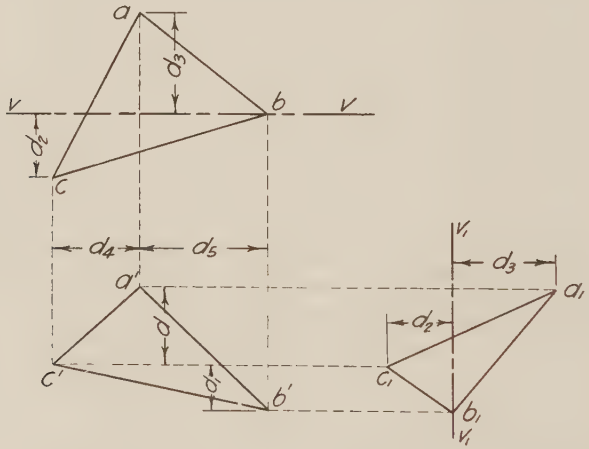


FIG. 12.—*Top, front, and right end views of triangle ABC.*

13. Picturing an object from its top view. Fig. 12 shows the top, front, and right end views of a triangle A B C. Any two of these views are sufficient to determine the size and form of the triangle and its position with respect to the picture planes. Place the drawing in a hori-

zontal position and picture the triangle from its top view *a b c*. Because the plane of the drawing is in the natural position of the horizontal plane, it is easier to picture from the top view than it is from the front or end views. Since we are dealing with orthographic projection, the point A is in a perpendicular to the paper through its top view *a*. Similarly B is in a perpendicular at *b* and C in a perpendicular at *c*. The relative distances of these points from the paper are shown in the front view. To get a definite picture of the triangle, suppose B coincides with its top view *b*. Then C is directly above *c* the distance d_1, obtained from the front view, and A is directly above *a* the distance $d+d_1$. This gives a definite picture of the triangle A B C and its position with respect to its top view. If A is pictured as coinciding with its top view *a*, then C is the distance *d* directly below *c* and B is the distance $d+d_1$ directly below *b*. This gives a definite picture of the triangle A B C and its position with respect to its top view. The triangle might have been pictured with C at *c*, A above *a*, and B below *b*.

14. Picturing an object from its front view. Picturing from the front view is easier from a drawing on the blackboard or when the drawing board is held in a vertical position, because the plane of the drawing in this position is in the natural position of the vertical picture plane. In Fig. 12, hold the plane of the drawing in a vertical position. The point A is considered to be in a perpendicular to the plane of the drawing at *a'*, B in a perpendicular through *b'*, and C in a perpendicular through *c'*. If B coincides with *b'*, then A is directly behind *a'* the distance d_3 and C is in front of *c'* the distance d_2. This gives a picture of the triangle A B C and its position with respect to its front view. The point C might have been pictured at *c'*, then A would have been behind *a'* and B behind *b'*. Or A might have been pictured at *a'*, then both B and C would have been in front of their respective front views. If the plane of

the drawing is placed in a horizontal position and the triangle pictured from its front view, distance which was back must now be thought of as down and distance which was toward the front must now be thought of as being above the plane of the drawing.

15. Picturing an object from its right end view. The plane upon which an end view is taken is perpendicular to the ground line and is called an *end* or *profile plane* or *P*, Fig. 6. The lines which project points upon the profile plane are parallel to the ground line and consequently show true length in both top and front views. In picturing the triangle A B C, Fig. 12, from its right end view, assume that B coincides with its end view b_1. Then A is the distance d_5 directly back of a_1 and C is the distance d_4+d_5 directly back of c_1. This gives a definite picture of the triangle A B C and its position with respect to the profile plane. This picture might have been formed by assuming A at a_1, then B would have been the distance d_5 directly in front of b_1, and C the distance d_4 back of c_1. Had C been pictured at c_1 then both A and B would have been directly in front of their respective end views.

16. Problems.

1. Given three points A, B, and C. (Fig. 1 below.)

(a) Place the drawing in a horizontal position. If A coincides with its top view a, where is B with respect to its top view b; where is C with respect to its top view c? If C coincides with its top view c, where are A and B with respect to their respective top views a and b?

(b) Place the drawing in a vertical position. If A coincides with its front view a', where is B with respect to its front view b'; where is C with respect to its front view c'? If B coincides with its front view b', where are A and C with respect to their respective front views a' and c'?

2. Given the three points X, Y, and Z. (Fig. 2 below.)

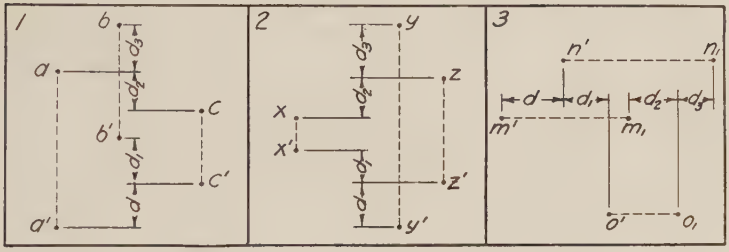
(a) Place the drawing in a horizontal position. If Z coincides with its top view z, where is Y with respect to its top view y; where is X with respect to its top view x? If X coincides with its top view x, where are Y and Z with respect to their respective top views y and z?

(b) Place the drawing in a vertical position. If Y coincides with its front view y', where is Z with respect to its front view z'; where is X with respect to its front view x'? If X coincides with its front view x', where are Y and Z with respect to their respective front views y' and z'?

3. Given the three points M, N, and O. (Fig. 3 below.)

(a) Place the drawing in a vertical position. If O coincides with its front view o', where is M with respect to its front view m'; where is N with respect to its front view n'? If M coincides with its front view m', where are N and O with respect to their respective front views m' and o'?

(b) Place the drawing in a profile position. If N coincides with its end view n_1, where is M with respect to its end view m_1; where is O with respect to its end view o_1? If M coincides with its end view m_1, where are N and O with respect to their respective end views n_1 and o_1?



AUXILIARY VIEWS

17. Auxiliary views. In this text, the term **auxiliary view** will be used for all except the top, front, or end view. If two views of an object, on planes which are at right angles to each other, are given, the form and position of the object are usually determined. As an example, the top and front views of the tetrahedron A B C E, Fig. 13, determine the form and position of the tetrahedron. With the top and front views given, any number of auxiliary views can be made easily.

18. Auxiliary views on planes perpendicular to H.

Let the top and front views of the tetrahedron* A B C E be given as in Fig. 13.

Analysis. Picture the four corners of the object directly over their top views as explained in Art. 13. The distances of the corners from a horizontal plane show true length in the front view and will also show true length on

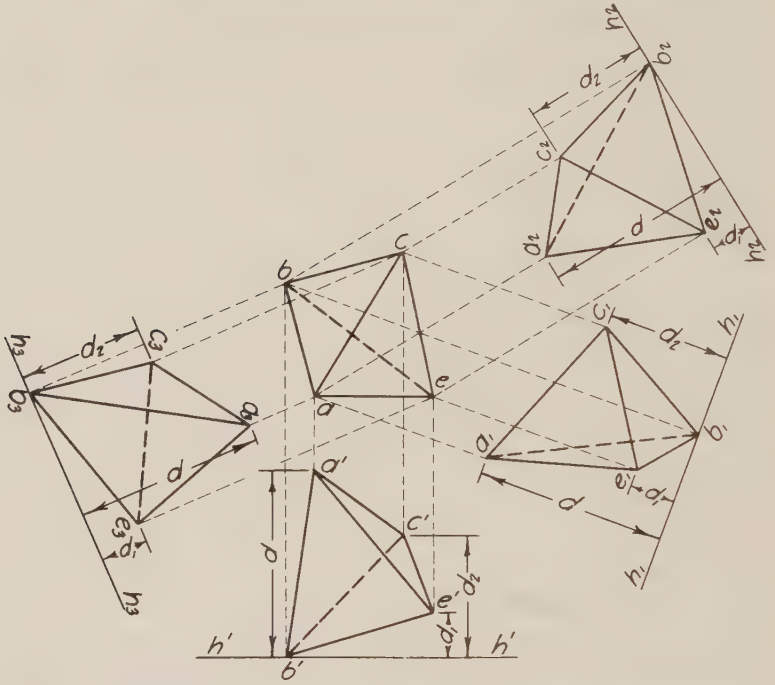


FIG. 13.—*Auxiliary views of tetrahedron on planes perpendicular to H.*

any picture plane perpendicular to H. With the top view of an object given and the heights of the corners of the object from a horizontal plane, the front view of the object can be made. In a similar manner the auxiliary view is made on any picture plane which is perpendicular to H. The width of the view on any of these picture planes is

* **A tetrahedron** is a solid bounded by four plane faces. It is a pyramid with a triangular base.

measured parallel to the line of intersection of this picture plane with H. The distance between perpendiculars drawn from the top views of the corners of the object to the picture plane is the width of the view on that plane.

Construction. Fig. 13 shows three auxiliary views taken on picture planes which are perpendicular to H. To construct the first auxiliary view, draw the line $h_1\ h_1$ at right angles to the desired line of sight and at a convenient distance from the top view. This represents a horizontal plane passing through the lowest corner B of the tetrahedron, $h'\ h'$ is the front view of this plane. Through the top views of the corners of the object, a, b, c, and e, draw lines perpendicular to $h_1\ h_1$. b_1 will lie on $h_1\ h_1$ since b' lies on $h'\ h'$. a_1, e_1, and c_1 are located by means of the distances d, d_1, and d_2 taken from the front view and measured from $h_1\ h_1$ toward the top view. Joining each of these points to the other three by straight lines completes the auxiliary view of the tetrahedron. The line $a_1\ b_1$ is dashed, since this edge of the tetrahedron is hidden in that view.

The next auxiliary view is constructed in a manner similar to the first. The line $h_2 h_2$, representing the horizontal plane through B, is drawn in the desired direction and any convenient distance from the top view. The vertical heights used are taken from the front view just as they were for the construction of the first auxiliary view. The edge A B is hidden in this view also.

The next auxiliary view is drawn in a manner similar to the first two auxiliary views. The line $h_3\ h_3$ is drawn in the desired direction, perpendiculars drawn to it from the top views of the corners of the object, and the auxiliary views of the corners located by means of distances taken from the front view. In this view, the edge C E is hidden.

19. Auxiliary views on planes perpendicular to V.

Let the top and front views of the square pyramid P—A B C E be given as in Fig. 14.

Analysis. Picture the corners of the pyramid from their front views as explained in Art. 14. The distances of the corners of the pyramid from a vertical plane show true length in the top view and will also show true length on any picture plane which is perpendicular to V. With the front view of an object and the relative distances of the corners of the object from the vertical plane given, the top view of the object can be made. In a similar manner, the auxiliary view of an object on any picture plane which is perpendicular to V can be drawn. Projecting lines are drawn from the front views of the corners of the object perpendicular to the line of intersection of the auxiliary

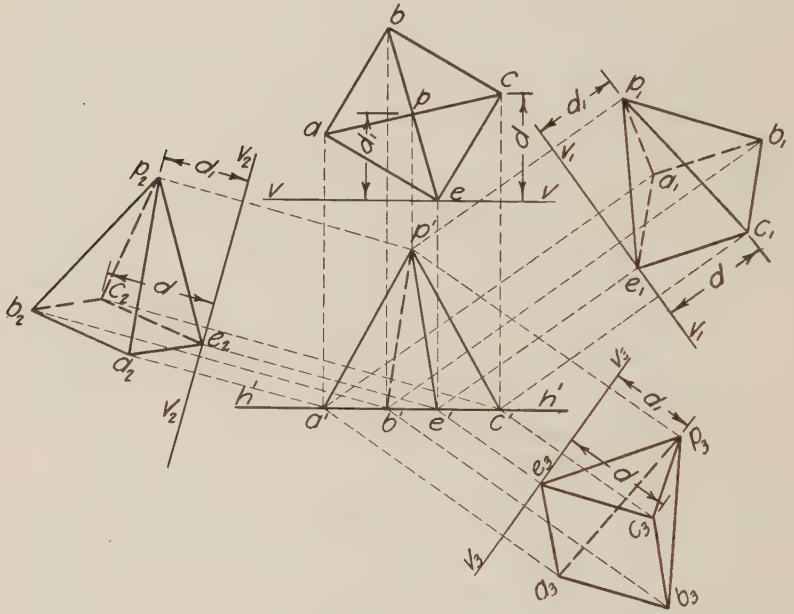


FIG. 14.—*Auxiliary views of pyramid on planes perpendicular to V.*

and vertical picture planes to construct the auxiliary view just as they are drawn perpendicular to the ground line in making the top from the front view.

Construction. Fig. 14 shows three auxiliary views taken on picture planes which are perpendicular to V. To

construct the first auxiliary view, draw the line $v_1\ v_1$ at right angles to the desired line of sight and at a convenient distance from the front view. This line represents a vertical plane through the corner E of the base, $v\ v$ is the top view of this plane. Through the front views of the corners of the object, draw lines perpendicular to $v_1\ v_1$. The point e_1 will lie on $v_1\ v_1$ since e lies on the line $v\ v$ in the top view. Since the point C is the distance d, shown in the top view, farther back than the vertical plane through E, the point c_1 will be located the distance d farther away from the front view than the line $v_1\ v_1$. In a similar manner, the other corners of the base and the vertex are located in the auxiliary view. Lines joining these points in the proper order complete the view of the pyramid. The corner of the base A is hidden in this view and consequently the edges running to this point are dashed.

The next auxiliary view is constructed in a manner similar to the first. The line $v_2\ v_2$, representing the vertical plane through E, is drawn in the desired direction and at a convenient distance from the front view. The distances the corners of the object are back of this plane are taken from the top view and used to locate the auxiliary views of the corners of the base and the vertex. C is the hidden corner in this view so the lines meeting at c_2 are dashed.

In constructing the third auxiliary view, the line $v_3\ v_3$ representing the vertical plane through E is drawn first. The distances d, d_1, etc., are taken from the top view and used to locate the respective corners of the object in the auxiliary view. In this view the base of the pyramid and vertex are visible but the lateral edge AP is hidden.

20. Auxiliary views on picture planes perpendicular to P. Let the top, front, and right end views of the tetrahedron ABCE be given as in Fig. 15.

Analysis. Picture the corners of the tetrahedron from their right end views as explained in Art. 15. The distances

of the corners of the tetrahedron from the profile plane P are shown in both the top and front views. These distances also show true length on all picture planes perpendicular to P. By projecting from the end views of the corners of the tetrahedron and using the distances of the corners from the profile plane P, the auxiliary views of the tetrahedron are made.

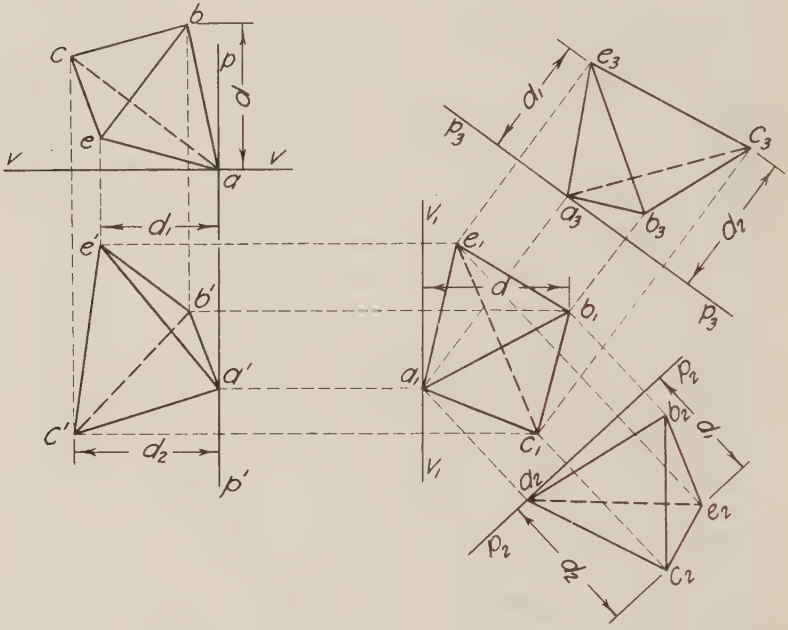


FIG. 15.—*Auxiliary views of tetrahedron on picture planes perpendicular to P.*

Construction. Fig. 15 shows two auxiliary views on picture planes which are perpendicular to P. To construct the first auxiliary view, draw the line $p_2\ p_2$ at right angles to the line of sight and at a convenient distance from the right end view. This line represents a profile plane P which is perpendicular to the ground line through A. Through the right end views of the corners of the object, draw lines perpendicular to $p_2\ p_2$. The point a_2 will lie on the line

$p_2\ p_2$ since a' lies on the line $p\ p'$. Since the point C is the distance d_2 from the profile plane P, the auxiliary view c_2 is the distance d_2 farther from the end view than $p_2\ p_2$. In a similar manner b_2 and e_2 are located. Joining these points by straight lines completes the view of the tetrahedron. The edge AE is hidden in this view.

To construct the next auxiliary view drawn $p_3\ p_3$ in the desired direction and any distance from the end view of the object. Draw perpendiculars to $p_3\ p_3$ from the end view of the corners of the object and locate the auxiliary view of the corners of the object on these lines. a_3 will lie on $p_3\ p_3$, e_3 is the distance d_1 farther away from the end view than the point a_3. In a similar manner b_3 and c_3 are located. In this view, the edge A C is hidden and consequently represented by a dashed line.

21. Auxiliary views on planes oblique to H, V, and P. In the front view, perpendicular distance from the corners of an object to H and P show true length; in the top view, distance from V and P show true length; in the end view, distance from H and V show true length. The preceding articles describe methods of drawing auxiliary views on picture planes perpendicular to H, V, and P with the use of these distances from H, V, and P. If, however, the picture plane upon which the desired view is to be taken is oblique to H, V, and P, these distances which show true length in the top, front, and end views will not show true length in the auxiliary view. Consequently the distances cannot be used as they were in preceding articles in making auxiliary views. The method of making an auxiliary view on a picture plane which is oblique to H, V, and P requires an intermediate auxiliary view and will be explained in Art. 33.

22. Relative position of views of an object on the drawing. In a working drawing, the top and front views of the object are usually shown. In space, the picture planes upon

which these views are made are at right angles to each other, one a horizontal and the other a vertical plane. Since both top and front views are made on one sheet of paper, it is evident that either H or V in its natural position with respect to the object must be thought of as being at right angles to the plane of the drawing. If, for example, the plane of the drawing be in a horizontal position and the object pictured directly above or below its top view, the vertical picture plane V should be pictured as perpendicular to the plane of the drawing. After the projection of the object is made on V, **this plane must be imagined taken from its natural position and laid with the side which is away from the point of sight on the drawing surface.** The front view of the object is usually placed lower on the sheet than the top view.

If the plane of the paper is thought of as V, then H must be pictured perpendicular to the plane of the drawing. If the plane of the paper is temporarily held in a vertical position and H perpendicular to it, the picture will seem more natural. After the projection of the object is made on **H, this plane must be imagined taken from its natural position and laid with the side which is away from the point of sight on the drawing surface.**

The above method is followed in placing auxiliary views on the drawing as shown in Figs. 13, 14, and 15. If auxiliary views are made on planes perpendicular to H, as in Fig. 13, the view is placed on the side of the top view on which the observer is stationed. In a similar manner, the auxiliary views in Figs. 14 and 15 are placed at the side of the given view on which the observer is stationed. **In all cases, the auxiliary picture plane is taken from its natural position and laid with the side which is away from the point of sight on the drawing surface.**

23. Problems.

1. A(—2, 2¼, —1), B(—1, 2¼, —2½), and C (0, 2¼, —2) are the three corners of the base of a triangular prism whose lateral edges are perpendicular to V. Construct an auxiliary view which will show the true size of the section cut from the prism by a plane which is perpendicular to H and 45° with V.

2. M(—1½, 0, —1¾) N(—1½, 2¼, —1¾) is the axis of a right circular cylinder, 1½″ in diameter. Cut this cylinder with a plane which contains the mid-point of MN, makes 45° with V, and is perpendicular to H. Draw the major and minor axes of this elliptical section in the view which shows the section true size.

3. O(0, 7/8, —2½) is the front corner of the base of a 1½″ cube. The base is horizontal and has one side 30° with V. Cut this cube with a plane which passes through its upper left hand and lower right hand corners and is perpendicular to V. Construct an auxiliary view which will show the true size of the section cut by this plane.

4. A 2″ square which lies in a horizontal plane with two sides parallel to the ground line is the base of a right pyramid. The vertex V (0, 1½, —2¾) of the pyramid is 2¾″ below the center of the square base. The pyramid is cut by a horizontal plane through the center of the altitude and the upper half of the pyramid is used as a sheet metal hopper. Construct an auxiliary view which shows the true size of one trapezoidal face of the hopper.

5. O (—2¾, 0, —1¾) is one corner of the base of a right square prism which has A(—2, 7/8, —2½) B(—2, 2⅛, —1) as an axis. Construct the right end view which will show the plane of the base as a line. Construct an auxiliary view from the end view which will show the base true size.

6. X(—2¾, 2¼, —2½), Y(—¾, 2¼, —2½), and Z(—1¾, ¾, —2½) are the three corners of the base of a right triangular prism whose lateral edges are perpendicular to H. Construct the right end view of this prism. Pass a plane through this prism which contains the side of the base XY, makes 45° with H, and is perpendicular to the profile plane. Construct an auxiliary view which will show the true size of the section cut from the prism by this plane.

CHAPTER II

STRAIGHT LINES

24. Views of a line. The word "line" will be taken to mean "straight line" unless statement to the contrary is made.

A line appears as a line in all views with the exception of the view taken by looking in the direction of the line itself; in this case it is a point.

At least two views of a line on planes which are not parallel to each other must be shown before the direction and length of the line are determined. It is *not* sufficient to show just the top view or just the front view, but both top and front views must be shown, or one of these with some other, such as the end view. If two views of a line are shown, other views can be derived from them.

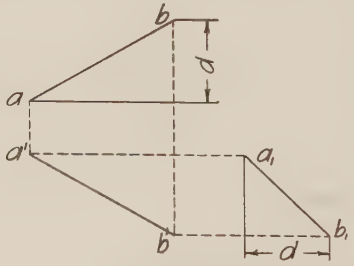


FIG. 16.—*Top, front, and right end views of line AB.*

Fig. 16 shows the top, front, and right end views of a line AB. The top view of the line is obtained by looking directly down from the top. This view shows that the line slopes backward to the right. The front view is obtained by looking directly from the front, and shows that the line slopes downward to the right. The right end view is obtained by looking from the right, in the direction of the ground line, and shows that the line slopes backward and

downward. d is the distance which B is farther back than A. This distance shows in the top view and also in the end view.

If the top and front views of the line AB are given, the right end view can be found as shown in Fig. 16. The distance d which B is farther back than A is taken from the top view and used to locate b_1 in the end view. The line perpendicular to G.L. through a_1 represents a vertical plane through the point A and is placed at any convenient distance to the right of the front view. Quite often in practical work, distances from front to back in the top view and from left to right in the end view are measured from the center line of the object.

25. Views of lines which are in various positions with reference to H, V, and P.

The position of a line in space can be described by giving its direction with respect to the principal planes of projection, H, V, and P.

A horizontal line is paralled to H. All points of the line have the same elevation. When viewed from the top, all points of the line are the same distance from the observer. There are any number of horizontal lines, and since they can make any angle with V, they may or may not be parallel to each other. The lines can be at different elevations.

A frontal line is parallel to V. When viewed from the front, all points of the line are the same distance from the observer. There are any number of frontal lines, and since they can make any angle with H, they may or may not be parallel to each other. The lines can be different distances from V.

A profile line is parallel to P. All points in a profile line are the same distance to the left or right. When viewed from the side, all points of the line are the same distance

from the observer. Profile lines need not be parallel to each other although their top views and also their front views are parallel.

An oblique line is not parallel to H, V, or P. No two points on an oblique line are the same elevation, the same distance to the left or right, or the same distance to the front or back.

26. Directions of views of lines with respect to G. L.

An oblique line has its top and front views oblique to G.L. In a similar manner describe the directions with respect to G.L. of the top and front views of the following lines:

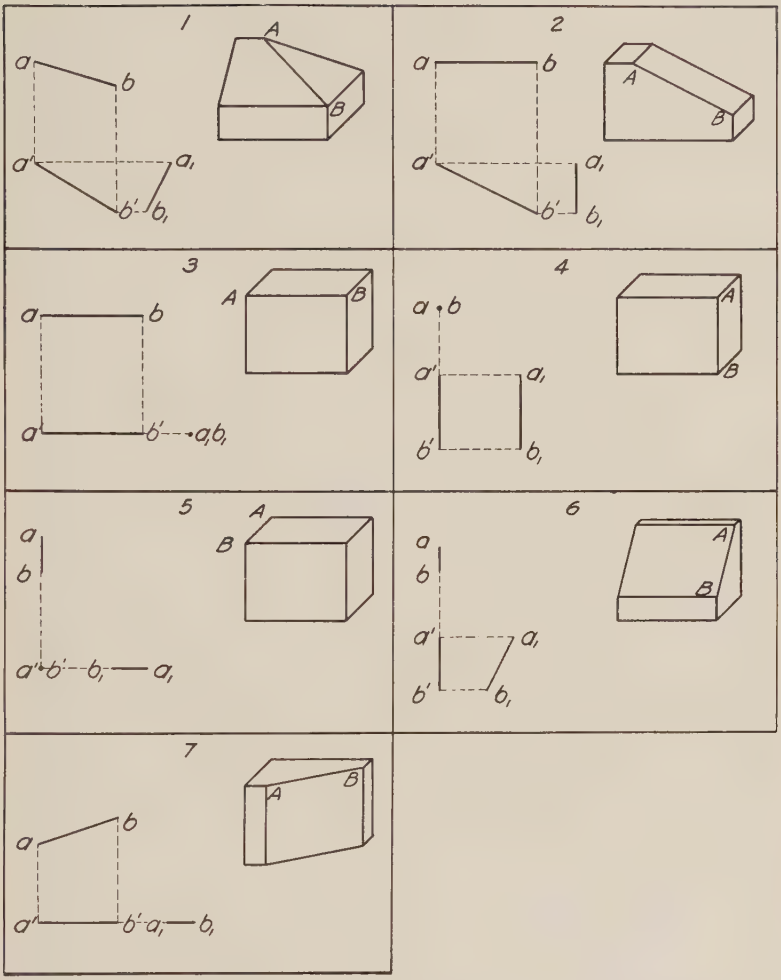
(a) A horizontal line which is oblique to V.
(b) A frontal line which is oblique to H.
(c) A profile line which is oblique to H and V.
(d) A line parallel to G.L.
(e) A line perpendicular to H.
(f) A line perpendicular to V.

Name the line AB as shown in Figs. 1-7 on page 29; for example, a horizontal line, a profile line, etc.

27. Point on line. If a point is on a line in space, the top view of the point is on the top view of the line, the front view of the point is on the front view of the line, etc., Fig. 17.

If the top and front views of a line AB are given, and also the top view of a point C on the line AB, the front view of C is found by drawing a perpendicular to G. L. through c and extending it to cut a' b' at c'. If the front view of C had been given the top view could have been located in a similar manner.

If the top and front views of a line are oblique to G. L., as AB, Fig. 17, the direction of the line is definitely determined without having particular points of the line lettered or designated. If, however, the top and front views of a



Top, front, and right end views of line AB in various positions with respect to H, V, and P.

line are perpendicular to G. L., as they are with a line of profile AB, Fig. 18, both views of two points of the line must be lettered before the direction of the line is fixed. For example, suppose the top and front views of a line of profile are given as in Fig. 18, but having the front view unlettered. If the upper end of the front view is lettered a' and the lower end b', a definite line is represented. If, however, the lower end is lettered a' and the upper end b', a line of different direction is represented. This is true

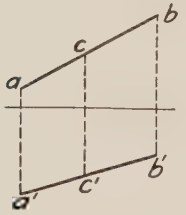


FIG. 17.—*Point C on line AB.*

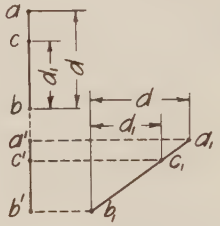


FIG. 18.—*Point C on line AB.*

since lines joining the top and front views of all points on a line of profile coincide. If the front view c' of a point on a given line of profile AB, Fig. 18, is represented, the top view of C cannot be found by the usual method of a perpendicular to G. L. The end view c_1 of the point is on the end view, a_1 b_1, of the line. From the end view of the point, its top view c is located. If the top view of a point on the line had been given, the front view of the point could have been determined in like manner from the end view.

28. **Intersecting lines.** If two lines intersect in space, Fig. 19, the line joining the point of intersection d of their top views with the point of intersection d' of their front views must be perpendicular to G. L. If CE, Fig. 19, had been a line of profile, an end or auxiliary view would have been necessary to prove that CE and AB intersect at D. The lines might appear from top and front views to intersect but the points C, D, and E may not be in a straight line.

Non-intersecting lines. Two non-intersecting lines, Fig. 20, may be in such a position that their top views and also their front views are intersecting. The line joining the point of intersection d of their top views with the point of intersection g' of their front views is not perpendicular to G. L.

Parallel lines. If two lines are parallel in space, Fig. 21, a view of the lines from any position, other than from a point on the plane of the lines, will show them as parallel lines. Hence any view of a parallelogram, from a point

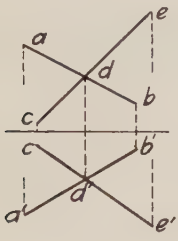


FIG. 19.—*Intersecting lines.*

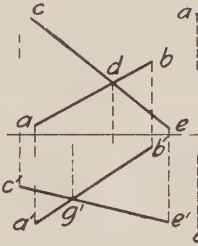


FIG. 20.—*Non-intersecting lines.*

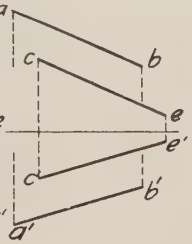


FIG. 21.—*Parallel lines.*

outside of the plane of the figure, is a parallelogram. If the lines AB and CE, Fig. 21, were both lines of profile, the most convenient proof that the lines are parallel to each other is an end or auxiliary view. The lines will appear parallel from top and front views and yet may not be parallel in space.

29. Problems.

1. A (—1½, 1, —½), B (—¾, 0, —1¾), and C (1¼, ¾, —2¼) are three corners of a parallelogram ABCE. Draw its top and front views.

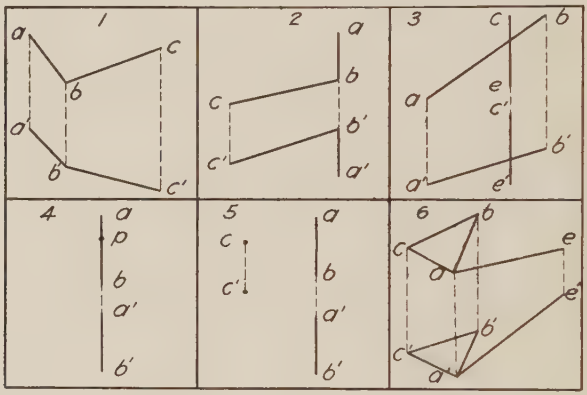
2. A (¼, 1½, —1½), B (¼, ½, —½) and C (—2, 0, —1¼) are three corners of a parallelogram ABCE. Draw its top, front, and right end views.

3. Given two lines, A (—2½, ½, —2¼) B (0, 2¼, —1½) and C (—¾, 2¼, —¾) E (—¾, ¾, —2¼). Do the lines intersect?

4. If P (0, 1½, ?) lies on the line A (0, 1¾, —¾) B (0, ¼, —2), locate its front view.

5. Locate the top and front views of a point E so that the line C (—2, 1¼, —½) E (?, ?, ?) will be parallel to the line A (—½, 1¾, —1) B (—½, ½, —2¼).

6. A (—1¼, ¼, —2¼) B (—¾, 1½, —1¼) C (—2¼, ¾, —1¾) is one base of a triangular prism having A E (1, ¾, —½) for a lateral edge. Draw the top and front views of the prism showing hidden edges dashed.



REVOLUTION AND COUNTER-REVOLUTION

30. An object is said to revolve about a straight line as an axis when each of its points moves in the circumference of a circle whose center is in the axis and whose plane is perpendicular to the axis.

When an object is revolved about a straight line as an axis, the relative position of its points is not changed. The object can thus be brought into a simpler position with reference to the planes of projection. The views of the object in this position are easily found and from these the views of the object in its original position are located by the counter-revolution of its points.

If a point revolves about an axis, a view of its path taken by looking in the direction of the axis is a circle. A view of its path taken by looking in a direction perpendicular to the axis is a straight line at right angles to the axis and equal in length to the diameter of the circle. A view from any other direction will show the path of the point as an ellipse.

If an axis of revolution is perpendicular to H, describe the top, front, and end views of the path of a point revolving about it. In a similar manner, describe the path of a point revolving about an axis which is perpendicular to V; one perpendicular to P.

31. To revolve a point about a horizontal axis.

Let P, Fig. 22, be the given point and SS the given axis.

Analysis. Since the axis is parallel to H, the plane of the circle in which the point moves is perpendicular to H. The top view of the circle is a straight line through the top view of the point perpendicular to the top view of the axis and equal in length to the diameter of the circle. If the horizontal axis is parallel to V, the end view of the path of the moving point is a circle; if the axis is perpendicular to V, the front view of the path is a circle; if oblique to V, an

auxiliary view must be taken to show the path as a circle. The elevations of points on the circumference of the circle are measured from a horizontal plane containing the axis and show true height in the front view and also in the view which shows the path of the point as a circle.

Construction. Picture the axis SS and the point P directly over their top views, Art. 13. If the axis coincides with its top view then P is the distance d directly above its top view p. The top view of the path of P is a straight line through p perpendicular to ss. The horizontal plane containing the axis has h' h' for a front and h_1 h_1 for an auxiliary view. h_1 h_1 is at right angles to s s extended and at a convenient distance from the top view. s_1 on h_1 is the auxiliary view of the axis of rotation. Through p, draw a line parallel to s s and with the distance d taken from the front view locate p_1 in the auxiliary view. Distance above the horizontal plane containing the axis is measured in the auxiliary view from h_1 h_1 toward the top view and distance below the horizontal plane is measured from h_1h_1 away from the top view. With s_1 as center and s_1 p_1 as radius draw the circle which is the auxiliary view of the path of the point. If the point is stopped at any position on the circle in the auxiliary view, its top and front views are found as follows: Through the auxiliary view of the point,

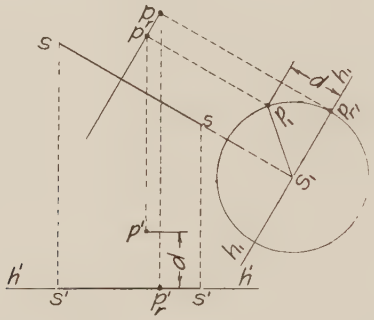


FIG. 22.—*Point P revolved about axis SS.*

draw a line parallel to *ss* until it intersects the top view of the circle, which is a line through p at right angles to *ss*. This intersection is the top view of the point. The front view of the point is directly below the top view, and the same distance above or below h' h' that its auxiliary view is above or below h_1 h_1. p_r and p_r' are the top and front views of the point when it is at the level of the axis SS.

32. To revolve a triangle about an axis which is parallel to V.

Let ABC, Fig. 23, be the given triangle and FF the given axis.

Analysis. Since the axis is parallel to V, the front views of the paths of the points A, B, and C are straight lines perpendicular to the front view of the axis. The auxiliary view in the direction of the axis shows these paths as circles and the axis as the center of the circles. After the auxiliary view of the triangle in its original position is drawn, the triangle can be revolved to any desired position in the auxiliary view. From the auxiliary view of the triangle in its new position, the front and then the top view can be derived.

Construction. Picture the axis FF and the points A, B, and C from their front views, Art. 14. If the plane of the paper is considered to be V and the axis coincides with its front view f' f', then A and B are back of or below a' and b' and C is in front of or above c'. The distances of A, B, and C from the plane of the paper are shown in the top view. The vertical plane containing the axis has v v for a top and v_1 v_1 for an auxiliary view. v_1 v_1 is at right angles to f' f' and at a convenient distance from the front view. f_1 is the auxiliary view of the axis. Through a', b', and c', draw lines parallel to f' f' and by means of distances as d taken from the top view, locate a_1, b_1, and c_1 in the auxiliary view. Distances in front of the vertical plane through the axis are measured in the auxiliary view from v_1 v_1 toward the front

view and those back of the vertical plane are measured from v_1 v_1 away from the front view. The auxiliary views of the paths of the points are circles with f_1 as center while the front views of these paths are straight lines through the front views of the respective points perpendicular to f' f'. Let the triangle ABC be rotated until the point A is ½″ farther toward the front than the vertical plane through the axis. a_{r_1} is first located on the circle through a_1 and ½″ farther toward the front view than v_1 v_1. From

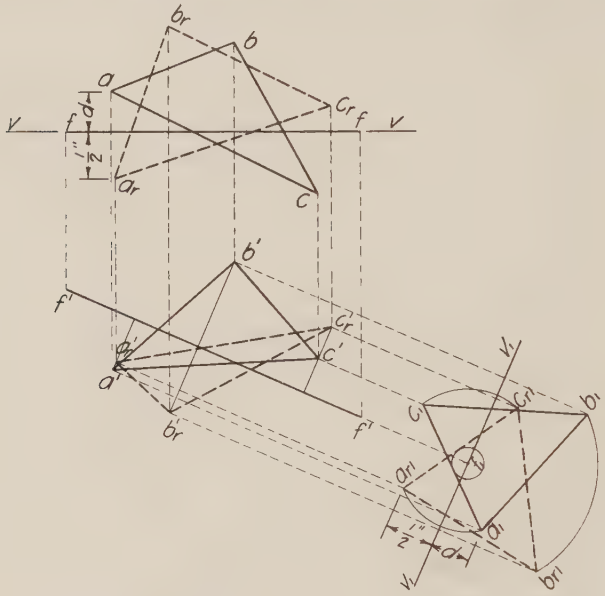


FIG. 23.—*Triangle ABC revolved about axis FF.*

a_{r_1} the front view a_r' is located and then the top view a_r. With f_1 as center draw a circle tangent to a_1 c_1. Through a_{r_1} draw a tangent to this circle and extend it to cut the circle through c_1 at c_{r_1}. This is the auxiliary view of the rotated position of the point C. c_r' and c_r are located in the same way that a_r' and a_r were located. In the auxiliary view, draw a circle with f_1 as center and tangent to c_1 b_1. From c_{r_1} draw a tangent to this circle and extend it to cut the circle through b_1 at b_{r_1}. b'_r and b_r are located in a

manner similar to that used to locate a_r' and a_r. $a_r\ b_r\ c_r$ is the top and $a_r'\ b_r'\ c_r'$ is the front view of the triangle ABC in its rotated position. The views of the triangle in its rotated position are shown in dashed lines to distinguish them from the views of the triangle in the original position. The triangle ABC could have been rotated to many other positions with F F as an axis and the auxiliary, front, and top views of the triangle found in its new position.

33. To revolve a point about an axis which is oblique to both H and V.

Let P, Fig. 24, be the given point and AB the given axis.

Analysis. The path of the point P revolving about the axis AB is a circle which appears as an ellipse in the top and front views. An auxiliary view which shows the axis AB true length will show the path of P as a straight line through the auxiliary view of P perpendicular to the auxiliary view of AB. A second auxiliary view which shows the axis AB as a point will show the path of P as a circle. Therefore to rotate conveniently the point P about AB as an axis, these two successive auxiliary views are necessary.

Construction. The axis AB will show true length in an auxiliary view if the picture plane of the view is parallel to the line AB. Picture the line AB in space over its top view ab, B coinciding with b and A the distance d directly above a. Place an auxiliary picture plane parallel to AB and perpendicular to H, and project A, B, and point P upon this plane. The projection lines from the top view to the auxiliary view will be perpendicular to the top view of AB. Draw lines through a, b, and p perpendicular to $a\ b$. At any convenient distance from the top view, select the point b_1 on its projection line, and construct $h_1\ h_1$ through this point parallel to $a\ b$. $h_1\ h_1$ is the auxiliary view of a horizontal plane which has $h'\ h'$ as its front view. a_1 and p_1 are located on their respective projection lines by means of

the distances d and d_1 taken from the front view. The circle traced by P as it revolves about AB will show as a straight line through p_1 and at right angles to a_1 b_1.

The axis AB will appear as a point on a picture plane which is perpendicular to AB. Picture AB and P in space over the first auxiiary view, A coinciding with a_1, B with

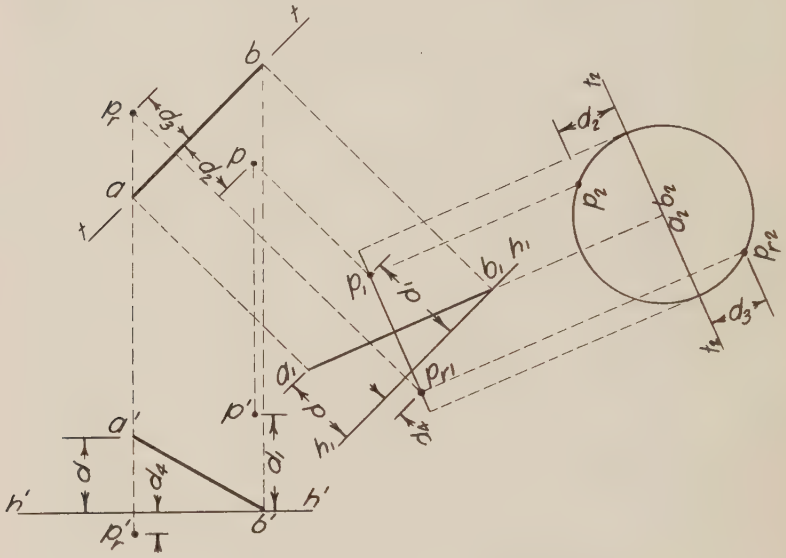


FIG. 24.—*Point P revolved about axis AB.*

b_1, and P the distance d_2 directly over p_1. Place a second auxiliary picture plane perpendicular to the line a_1 b_1, and project A, B, and point P upon this plane. The projection lines from the first to the second auxiliary view will be parallel to a_1 b_1. At a convenient distance from the first auxiliary view, select a_2 on its projection line, and draw t_2 t_2 perpendicular to a_1 b_1. t_2 t_2 is the second auxiliary view of a plane which contains the line AB and is perpendicular to H, t t is the top view of this plane. b_2 and p_2 are located on their respective projection lines, the distances from the plane t_2 t_2 being determined from the top view. The path of P as it revolves about AB will appear as a circle in this view.

If the point P is stopped at any position on the circle, as p_{r2}, the top and front views of this point are found as follows: Through p_{r2} draw a line parallel to $a_1\ b_1$ until it intersects the first auxiliary view of the circle at p_{r1}. Project p_{r1} to the top view and by means of the distance d_3 taken from the second auxiliary view locate the top view p_r. The front view p_r' is directly below the top view p_r and the distance d_4 taken from the first auxiliary view, below $h'h'$.

34. Problems.

1. Rotate the point P (—¾, 1½, —⅞) about S (—2⅜, 1¾, —1½) S (¼, ¼, —1½) as an axis until it reaches the level of SS. Letter the top and front views of the point in this position.

2. In problem 1, rotate P until it is ½" lower than SS and letter its top and front views in this position.

3. Rotate the point P (1¼, ⅜, —1) through an angle of 60° about F (—¼, 1, —¾) F (2⅜, 1, —2¼) as an axis. Letter the top and front views of the point in its final position.

4. Revolve the line A (—¼, 1¾, —¾) B (—1¾, ½, —2¼) about S (—2⅜, 1¾, —1¼) S (¼, ¼, —1¼) as an axis until A is ⅛" lower than SS. Draw the top and front views of the line in this position.

5. Revolve the line A (—¼, 1¼, —¾) B (—1⅞, ½, —1¾) about S (—2⅜, 1¾, —1½) S (¼, ¼, —1½) as an axis until A is in the same horizontal plane with SS. Draw the top and front views of AB in this position.

6. Revolve the line A (—¼, 1¾, —1) B (—2, ½, —2½) about S (—2⅜, 1¾, —1½) S (¼, ¼, —1½) as an axis until B is ½" higher than SS. Draw the top and front views of AB in this position.

7. Revolve the line A (2, 1¼, —1) B (½, 0, —2) about F (—¼, 1, —¾) F (2⅜, 1, —2¼) as an axis until B is in the same vertical plane with F F. Draw the top and front views of A B in this position.

8. Revolve the line A (2, 1½, —1⅛) B (½, 0, —1¾) about F (—¼, 1, —¾) F (2⅜, 1, —2¼) as an axis until A and B are the same distance in front of a vertical plane through the axis. Draw the top and front views of A B in this position.

9. Rotate the triangle A (—2, ¼, —1) B (—¼, 0, —2¼) C (—1, 1¼, —1) about the line S(—2¼, 1¼, —1½) S (¼, ¼, —1½) as an axis until A comes into its lowest position. Represent the top and front views of the triangle in this position.

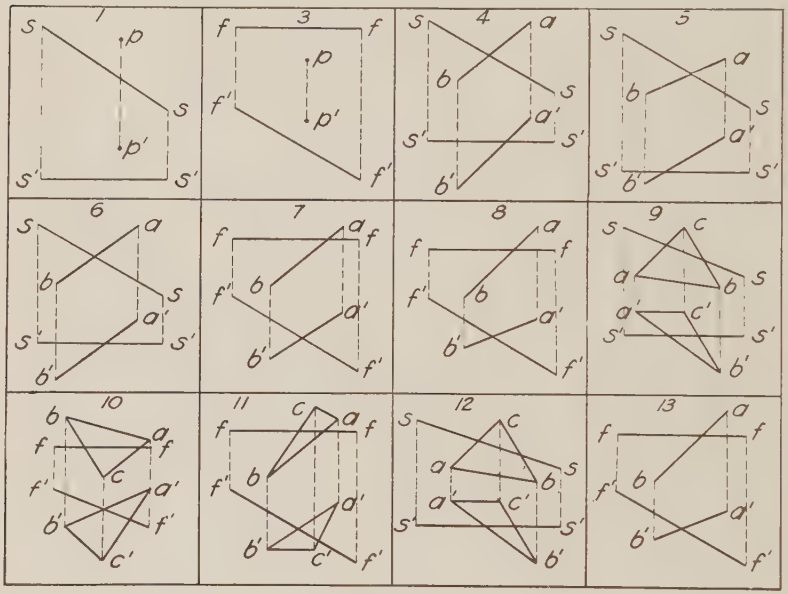
10. Rotate the triangle A (—¾, 1½, 0) B (—2½, 2, —¾) C (—1¾, ¾, —1½) about the line F(—2¾, 1⅜, 0) F(—½, 1⅜, —¾) as an axis until the point C is 1⅛" in front of the vertical plane containing FF. Represent the top and front views of the triangle in this position.

11. Rotate the triangle A (2, 1¼, —1) B (½, 0, —2) C (1½, 1½, —2) about the line F (—¼, 1, —¾) F (2⅜, 1, —2¼) as an axis until the point A is as far as possible behind the vertical plane containing the axis. Show the top and front views of the triangle in this position.

12. Rotate the triangle A (—2, ¼, —1) B (—¼, 0, —2¼) C (—1, 1¼, —1) about the line S (—2¼, 1¼, —1½) S (¼, ¼, —1½) as an axis until A is as far as possible to the front. Show top and front views of triangle in this position.

13. Revolve the line A (2, 1½, —1⅛) B (½, 0, —1¾) about F (—¼, 1, —¾) F (2⅜, 1, —2¼) as an axis until B is in its lowest position. Construct the top and front views of the line in this position.

14. Rotate the point O (—2¾, 1½, —2⅜) about A (—2½, ½, —¾) B (—1, 2½, —2½) as an axis until it is in the horizontal plane with the point A. Locate its top and front views in this position.



THE TRUE LENGTH OF A LINE

FIRST METHOD—AUXILIARY VIEWS

35. In orthographic projection, two views of a line will determine its direction in space, providing the planes upon which these views are taken are not parallel to each other or the line is not perpendicular to the line of intersection of the picture planes. If two points of the line are designated, the line will have a definite length between these points. Any view in which the line of sight is perpendicular to the given line will show the given line in its true length. If two views of a line are given, any number of other views can be derived from them. **In order conveniently to make an auxiliary view from two given views, the line of sight must be parallel to the plane upon which one of the given views is taken.** Therefore to find the length of a line when two of its views are given, take an auxiliary view with the line of sight perpendicular to the given line and parallel to the plane of one of the given views.

36. Given the top and front views of a line. Find the true length of the line and the angle it makes with H.

Let AB, Fig. 25, be the given line.

Analysis. Picture the line from its top view. If A coincides with its top view a then B is the distance d directly below its top view b. The angle which a line makes with H is the angle between the line and its top view. A view taken by looking at right angles to a plane containing the line and its top view will show the true length of the line and the angle which the line makes with H. In this case the line of sight will be parallel to H and perpendicular to the given line.

Construction. In Fig. 25, $h'h'$ is the front view of a horizontal plane which passes through A. Take an auxiliary view by looking at right angles to the plane abB. To do this draw lines at right angles to ab at the points a and b.

At any point in the line through a, select the point a_1, and draw $h_1\ h_1$ through this point parallel to ab. $h_1\ h_1$ is the auxiliary view of the horizontal plane which has $h'\ h'$ for a front view. b_1 is located on the line through b at right angles to ab and the distance d below $h_1\ h_1$. Then $a_1\ b_1$ is the true length of AB and Θ_H the angle AB makes with H.

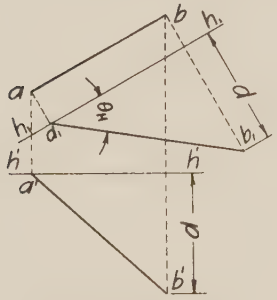


FIG. 25.—θ_H *angle AB makes with H.*

If AB had been parallel to V, the plane abB would have been parallel to V, and the front view would have shown the true length of the line and its angle with H. If AB had been a line of profile, the end view would have shown its true length and its angles with H and V.

37. If the top view ab of the line and the angle Θ_H which the line AB makes with H are given, the front view of the line can be found. With the top view of the line and the angle which the line makes with H given, the auxiliary view as shown in Fig. 25 can be constructed. The line $h_1\ h_1$ is the auxiliary view and the line $h'\ h'$ is the front view of a horizontal plane. These lines may be drawn at any convenient distance from the top view but $h_1\ h_1$ must be parallel to $a\ b$ and $h'\ h'$ must be parallel to G. L. Locate a_1 on $h_1\ h_1$ in the auxiliary view and draw the line $a_1\ b_1$ making Θ_H with $h_1\ h_1$, thus locating b_1. In the front view, a' is on $h'h'$ and b' the distance d below $h'h'$. This gives the required front view $a'b'$. The point B could have been placed on the horizontal plane and A the distance d below this plane and the line AB would satisfy the given conditions.

If the top view and true length of AB are given, the front view can be found. In the auxiliary view the point b_1 is located by means of the true length $a_1\ b_1$ rather than by the angle Θ_H and the front view $a'\ b'$ located by means of the distance d taken from the auxiliary view. There are two solutions, either A or B being the higher point.

If the front view a' b' and the angle Θ_H are given, the top view a b can be drawn. In this case you cannot tell at first where to place the auxiliary view because you do not know where to draw the line h_1 h_1 Consequently, draw h_1 h_1 in any position and with the aid of Θ_H and the distance d taken from the front view, construct the auxiliary view. This view will show the length a b of the top view. Draw a perpendicular to G. L. from a' and locate a at a convenient place on this perpendicular. With a as center and the length a b taken from the auxiliary view as radius, strike an arc cutting the perpendicular to G. L. through b' at the point b. Then a b is the required top view. Since the arc will cut the perpendicular to G. L. through b' in two places, there are two solutions to the problem.

38. **Given the top and front views of a line. Find the true length of the line and the angle which it makes with V.**

Let AB, Fig. 26, be the given line.

Analysis. Picture the line from its front view. If A coincides with its front view a' then B is the distance d directly back of its front view b'. The angle which a line makes with V is the angle between the line and its front view. A view taken by looking at right angles to a plane containing the line and its front view will show the true length of the line and the angle which it makes with V. In this case the line of sight will be parallel to V and perpendicular to the given line.

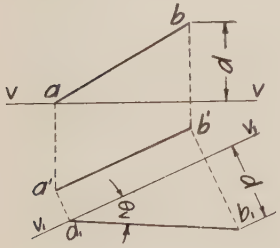


FIG. 26.—θ_v *Angle AB makes with V.*

Construction. In Fig. 26, vv is the top view of a vertical plane which passes through A. If the drawing is in a horizontal position, distance back is pictured as distance below the front view. Take an auxiliary view by looking

at right angles to the plane a' b' B. To do this draw lines at right angles to a' b' at the points a' and b'. At any point in the line through a', select the point a_1 and draw v_1 v_1 through this point parallel to a' b'. v_1 v_1 is the auxiliary view of the vertical plane which has vv for a top view. b_1 is located on the line through b' at right angles to a' b' and the distance d below v_1 v_1. Then a_1 b_1 is the true length of AB and Θ_v the angle which AB makes with V.

If AB had been parallel to H, the plane a' b' B would have been parallel to H, and the top view would have shown the true length of the line and its angle with V.

If the front view of the line is given and the angle which the line makes with V, the top view can be drawn. From the front view and the angle which the line makes with V, draw the auxiliary view, Fig. 26. This view shows the distance d which one end of the line is further back than the other. With this distance, the top view can be determined. There are two solutions to the problem.

If the front view and the true length of the line are given, the top view can be found. From the front view and the true length of the line construct the auxiliary view. This view shows the distance d which one end of the line is farther back than the other. With this distance, the top view can be determined. There are two solutions to the problem.

If the top view of the line and the angle Θ_v which the line makes with V are given, the front view can be drawn. In this case the position of the auxiliary view cannot be determined because the position of the line v_1 v_1 is not known. Consequently, draw v_1 v_1 in any position and with the aid of Θ_v and the distance d taken from the top view, construct the auxiliary view. This view shows the length $a'b'$ of the front view. With this length as a radius, the front view $a'b'$ in its proper position below the top view

can be constructed. There are two solutions since either end of the line may be the lower.

39. Application of the first method for finding the true length of a line to the development of the surface of a prism.

By the development of a given surface, such as the surface of a prism or cone, is meant a plane area of such a form and size that it can be rolled or folded into the form of the given surface.

Let the prism be given as in Fig. 27.

Analysis. Take a section of the prism by a plane at right angles to the lateral edges. This section becomes a straight line on the development and the width of the faces from one lateral edge to the next can be measured along it. The lateral edges are drawn at right angles to this section line in the development, and the distances from the plane of the section to the upper and lower bases are laid off on the respective edges in the development. Straight lines joining in the proper order the ends of the lateral edges complete the development of the prism. The bases of the prism in the development can be attached to the ends of any one of the lateral faces.

Construction. Picture the prism in space over its top view. Construct an auxiliary view of the prism which will show the true lengths of the lateral edges, as b_1y_1, a_1x_1, c_1z_1. The distance d is taken from the front view and used to locate y_1 in the auxiliary view. The plane cutting at right angles to the edges of the prism appears as a line in the auxiliary view cutting the edges at m_1, n_1, and o_1. This plane can be placed at any convenient position along the edge of the prism.

Picture the prism in space over the auxiliary view. If the edge CZ coincides with its auxiliary view c_1z_1, then the edge AX should be pictured as being the distance d_1 directly below its auxiliary view a_1x_1. A second auxiliary view

taken in the direction of the lateral edges will show the prism as the triangle $m_2n_2o_2$. This triangle shows the

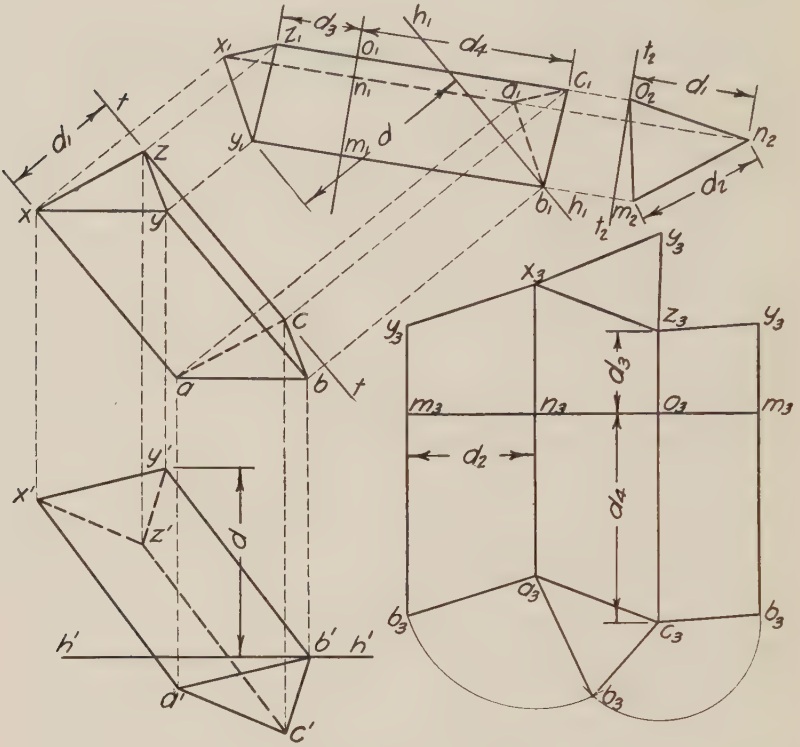


FIG. 27.—*Development of an oblique triangular prism.*

true size of the section of the prism along the line m_1o_1 of the first auxiliary view.

To construct the development of the prism, lay off on a straight line the three sides of the triangle $m_2n_2o_2$; $m_3n_3 = m_2n_2$, $n_3o_3 = n_2o_2$, and $o_3m_3 = o_2m_2$. At the points m_3, n_3, o_3 and m_3, erect perpendiculars to m_3m_3, and lay off on these lines the lengths of the edges taken from the first auxiliary view, above and below the plane of the section. This locates the ends of the lateral edges b_3y_3, a_3x_3, c_3z_3. Joining these points by straight lines completes the development of the

lateral surface of the prism. The lower base of the prism is a triangle with sides a_3b_3, a_3c_3, and c_3b_3 and the upper base a triangle with sides y_3x_3, x_3z_3, and z_3y_3.

Should the number of faces of the prism be increased indefinitely, the prism in the limit becomes a cylinder. Consequently, the method described above for developing a prism, is the method for developing the surface of a cylinder, see Art. 166.

40. Problems.

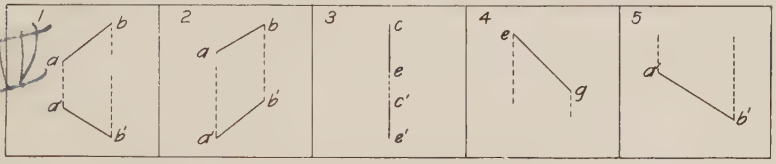
1. Find the true length of the line A (—¾, ¼, —1¼) B (½, 1½, —2¼) and the angle which it makes with H.

2. Find the true length of the line A (—¾, ¾, —2) B (½, 1¾, —¾) and the angle which it makes with V. Show the top and front views of a point P which is 1″ from A and on the line AB.

3. Find the true length of the line C (0, 1¾, —½) E (0, ¼, —2¼) and the angles which it makes with H and V.

4. Draw the front view of the line E (—½, 1¾, ?) G (1, ¼, ?) when its true length is 2½″.

5. Draw the top view of the line A (—½, ?, —¼) B (1½, ?, —1½) when the line makes 30° with V.



6. Draw the top and front views of the line A (—¾, ¼, —¾) B (¾, ?, —2) when the line makes 30° with H.

7. Find the angle which the line A (—2¾, ¾, —1) B (—¾, 1¾, —2) makes with the profile plane.

8. A drill hole which enters the earth at O (—1¼, ¾, —¼), bears N 30° E, dips 60°, and is 250′ long (scale 1″=100′). Assuming the G. L. runs east and west, draw the top and front views of the drill hole.

9. A (—5, 1½, —3½), B (—4¼, ¼, —3½), and C (—3, 1, —3½) are three corners of a square which is the base of an oblique prism having A (—5, 1½, —3½) X (—4, 3, —1¼) for a lateral edge. Draw the development of the lateral surface of the prism. (G. L. through the center and parallel to the longer side of a 7½″x11″ rectangle.)

10. A triangle A (—5, 1¼, —3½) B (—4½, 0, —3½) C (—3¼, ¾, —3½) is the base of an oblique prism. The top view of the edges makes 60° with G. L. The edges are 3½″ long and make 45° with H. Develop the surface of the prism. (G. L. through the center and parallel to the longer side of a 7½″ x 11″ rectangle).

11. A 1″ square stick has its edges making 30° to V and oblique to H. Find the true size of the section of this stick by a plane parallel to V; also by a plane parallel to P.

SECOND METHOD FOR FINDING THE TRUE LENGTH OF A LINE—ROTATION

41. Given the top and front views of a line, find the true length of the line and the angle which it makes with H.

Let AB, Fig. 28, be the given line.

Analysis. Rotate the given line about an axis which cuts it and is perpendicular to H until the line is parallel to V. In this position, the front view will show the true length of the line and the angle the line makes with H.

Construction. In Fig. 28, ac is the top and $a'\ c'$ the front view of an axis perpendicular to H. Over the top view, picture A at a, B directly below b, and AC perpendicular to the paper at $a\ c$. When AB is rotated about AC as an axis, A remains fixed and B moves in the arc of a circle to B_r, ab_r being drawn parallel to G. L. The top view

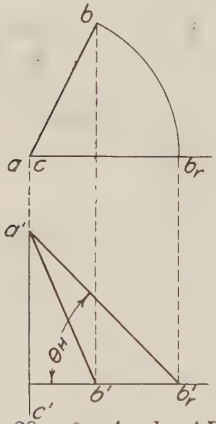


FIG. 28.—θ_H *Angle AB makes with H.*

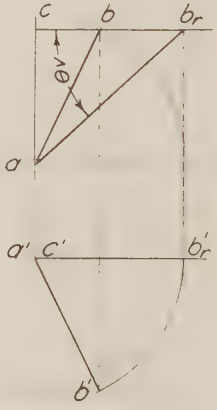


FIG. 29.—θ_V *Angle AB makes with V.*

of the path in which B moves is the arc bb_r and the front view the line $b'\ b_r'$ parallel to G. L. Since AB_r is parallel to V, the front view $a'\ b_r'$ shows the true length of the line AB. Θ_H is the true size of the angle AB makes with H.

42. Given the top and front views of a line, find the true length of the line and the angle which it makes with V.

Let AB, Fig. 29, be the given line.

Analysis. Rotate the given line about an axis which cuts it and is perpendicular to V until the line is parallel to H. In this position, the top view will show the true length of the line and the angle which the line makes with V.

Construction. In Fig. 29, a' c' is the front and ac the top view of an axis perpendicular to V. From the front view, picture A at a', B directly back of b', and AC perpendicular to the paper at $a'c'$. When AB is rotated about AC as an axis, A remains fixed and B moves in the arc of a circle to B_r, a' b_r' being drawn parallel to G. L. The front view of the path in which B moves is the arc b' b_r', and the top view the line bb_r parallel to G. L. Since AB_r is parallel to H, the top view ab_r shows the true length of line AB. Θ_v is the true size of the angle which AB makes with V.

43. With the help of Fig. 28 and Fig. 29, let the student solve the following problems by the rotation method.

Given the top view of a line AB and the angle Θ_H to find the front view of the line.

Given the top view of a line AB and the true length of AB to find the front view of the line.

Given the front view of a line AB and the angle Θ_H to find the top view of the line.

Given the front view of a line AB and the angle Θ_v to find the top view of the line.

Given the front view of a line AB and the true length of AB to find the top view of the line.

Given the top view of AB and the angle Θ_v to find the front view of the line.

44. Application of the second method for finding the true length of a straight line to the development of the surface of a pyramid.

Let the pyramid be given as in Fig. 30.

Analysis. Find the true lengths of the lateral edges and the sides of the base of the pyramid. This gives the true lengths of the three sides of each triangular face. Lay out the triangular faces true size, using the true lengths of the three sides of each triangle as radii.

Construction. Picture the pyramid in space over its top view. The lateral edges of the pyramid are all oblique to

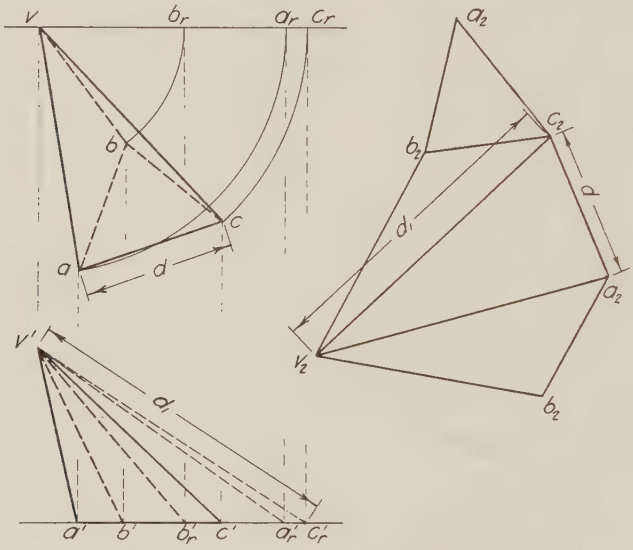


FIG. 30.—*Development of surface of pyramid.*

H and V and therefore are not true length in either top or front view. If these edges are rotated into the vertical plane about an axis which passes through V and is perpendicular to H, they will show true length in the front view. $v'a'_r$, $v'b'_r$ and $v'c'_r$ are the true lengths of these edges found in this manner. The base ABC is in a horizontal plane and consequently shows true size in the top view. If

the base were not horizontal, the true lengths of AB, BC, and AC would also be found by rotation. To find the true length of the lateral edges and the sides of the base, if necessary, by the auxiliary view method would require more construction than by the rotating method.

To make the development, select any point as v_2 and draw a line v_2a_2 in any direction through it equal in length to $v'a_r'$. With v_2 as a center draw an arc with radius $v'c_r'$. and with a_2 as a center draw an arc with $a\ c$ as a radius. These arcs intersect at c_2. This completes the first face of the pyramid. The other two faces are constructed in a similar manner. The base $a_2\ b_2\ c_2$ can be joined to any of the lateral faces in the development. It is a triangle constructed by arcs in a manner similar to that used in constructing the lateral faces of the pyramid.

45. Given the true length of a line and the angles which the line makes with H and V, to find the top and front views of the line.

Let the line be two inches long and make 30° with H and 45° with V, Fig. 31.

Analysis. Select any point A to be one end of the required line AB. Since AB is to make 30° with H it must be an element of a right circular cone with vertex at A and elements making 30° with H. Since AB is to make 45° with V it must be an element of a right circular cone with vertex at A and elements making 45° with V. Since AB is to make 30° with H and also 45° with V it must be an element of each of these cones and consequently their line of intersection. The cones having a common vertex will intersect in two straight lines, elements of both cones, and both being solutions to the given problem.

If the given angles are complementary, these cones with vertices at A will be tangent to each other. In this case there is one solution to the problem, the element of contact of the cones, and this solution is a line of profile.

If the sum of the given angles which the line is to make with H and V is greater than 90°, the two cones with vertices A will not touch each other except at the vertex. Consequently the cones do not have common elements and the problem is impossible of solution.

Construction. In Fig. 31, select a and a' as the top and front views of the point A in any convenient position. Draw $a'b'_{r1}$ 30° with G. L. and lay it off two inches long. Draw ab_{r1} parallel to G. L. and project b'_{r1} to the top view, locating b_{r1}. ab_{r1} is the radius of the base of the cone with

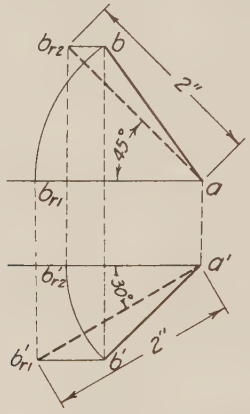


FIG. 31.—*Line AB makes 30° with H and 45° with V.*

vertex A and elements 30° with H. With a as center and ab_{r1} as radius draw the arc $b_{r1}b$, the top view of part of the base of the cone; b'_{r1} b' parallel to G. L. is the front view of part of the base. To construct the views of the other cone, draw ab_{r2} 45° with G. L. and two inches long. $a'b'_{r2}$ is the front view of the radius of the base. In the front view, the part of the base of this cone $b'_{r2}b'$ intersects the front view $b'_{r1}b'$ of the part of the base of the other cone at the point b'. The top views of the bases of these cones intersect at the point b. Then a b is the top and $a'b'$ the front view of the required line which makes 30° with H and 45° with V. The bases of the cones would intersect again to the right of A, giving a second solution to the problem.

Had the angles with H and V been given as complementary, say 30° and 60°, the bases of the cones would have been tangent at the point B. Then both $a'b'$ and $a\ b$ would have been perpendicular to G. L. and there would have been but one solution to the problem.

Had the sum of the angles with H and V been greater than 90°, neither the front nor top views of the bases of the cones would have intersected. Consequently the sum of the angles which a line makes with H and V cannot be greater than 90°.

46. Problems.

1. Draw the front view of the line A (—1¼, ¼, —⅛) B (½, 1¾, ?) when the line makes 30° with V.

2. A (—1½, ½, 0) B (1, ?, —2) is a line which makes 30° with H. Draw the top and front views of AB.

3. Given the tetrahedron A (—3, 1¼, —3¼) B (½, 2, —3¾) C (—2, 3¾, —4) E (—½, 4, —½). Make a paper model of the tetrahedron. (Use a 7½″ x 11″ rectangle).

4. The three guy wires of a derrick run from the upper end of the mast O (0′, 12′,—6′) to points of support at A (—22′, 3′, —15′), B (14′, 34′, —23′), and C (24′, 2′, —19′). What is the maximum length of a boom fastened 3′ from the lower end of a 30′ mast which will clear all the guy wires? How long is each guy wire? (Scale ⅛″=1′ —0″. Use a 7½″x11″ rectangle. The boom is fastened to the mast in such a manner that it can swing around the mast and also swing up and down about the point of support.

5. An observer stationed on a promontory at A (0, ½, —½), 1600′ above sea level, sighted a ship at 10 A. M. and recorded its location as N 30° E, with an angle of depression of 30°. At 10:03 A. M. the ship was observed from the same station at a bearing of N 30° W and at an angle of depression of 45°. Neglecting the curvature of the Earth, and assuming that the ship continued in the same straight course without change of speed, what was the bearing of the ship from the promontory, and what was the angle of depression at 10:06 A. M? What was the speed of the ship? (Scale 1″=1600′).

6. A (—1, ¼, —¼) B (?, ?, ?) is a line which makes 45° with H, 30° with V, and is 2¼″ long. Draw the top and front views of the line AB.

G. L. through the center and parallel to the longer side of a 7½″ x 11″ rectangle for problems 7 and 8.

7. A (—3¾, 1½, —3½), B (—3¼, 0, —3½), and C (—1¾, ½, —3½) are three corners of a square which is the base of an oblique pyramid with vertex at V (—1, 3, —½). Draw the development of the lateral surface of the pyramid.

8. Given a tetrahedron O (—1, 3, 0) A (—3, 2½, —3) B (—4, 1, —1½) C (—2, ¼, —2½). Select a point on one of the edges and then draw the top and front views of the shortest path to be followed on the surface of the tetrahedron, beginning at this point, crossing three of the faces and returning to the same point.

CHAPTER III

PLANES — LINE AND PLANE RELATIONS

47. **A plane** is a surface such that a straight line joining any two points in it lies wholly in the surface. A plane is understood to be indefinite in extent.

A straight line which is parallel to a straight line on a plane is parallel to the plane.

The intersections of two parallel planes by a third plane are parallel lines.

A line which is parallel to each of two interesting planes is parallel to their line of intersection.

A plane parallel to the line of intersection of two planes will cut parallel lines from the planes.

A triangle is a portion of a plane bounded by three straight lines.

A triangle is called a **scalene triangle** when no two of its sides are equal; an **isosceles triangle,** when two of its sides are equal; an **equilateral triangle,** when its three sides are equal.

A quadrilateral is a portion of a **plane** bounded by four straight lines.

A parallelogram is a quadrilateral which has its opposite sides parallel.

A rectangle is a parallelogram which has its angles right angles.

A polygon is a portion of a **plane** bounded by straight lines. **A regular polygon** is a polygon which is both equilateral and equiangular.

Two straight lines are **perpendicular** to each other, (1) when they are intersecting and form right angles with each other; or (2) when they are non-intersecting and yet are

in such positions that if one be moved parallel to itself until it cuts the other, right angles are formed.

A straight line is **perpendicular to a plane** when it is perpendicular to every straight line on the plane.

48. A plane must usually be considered as indefinite in extent. Points or lines may be drawn on the plane, but do not bound it. In space, a plane is fixed or represented by three points, not in the same straight line, a point and a line, two intersecting lines, or two parallel lines. If the top and front views of these magnitudes are given, the plane which they determine will have its inclination with H and V definitely fixed.

The points A, B, and C, Fig. 32, represent a plane in space. It is evident that a line joining A and B, B and C, or A and C will lie in this plane.

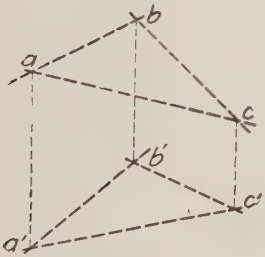


FIG. 32.—*Points A, B, and C represent a plane.*

If a plane is represented by a point and a line, other lines of the plane can be found by joining the given point with points on the given line.

If the plane is represented by two lines, either parallel or intersecting, any number of other lines of the plane can be found by joining a point in one of the given lines with a point in the other. When the lines are intersecting, Fig. 19, the line which joins the top and front views of their point of intersection must be at right angles to the ground line. If the lines are parallel, Fig. 21, all views of the lines will show them parallel. If the lines are neither parallel nor intersecting, Fig. 20, they do not represent a plane.

The top views of all lines of a plane which is perpendicular to H will all lie in one straight line. The front views of these lines will be either parallel or intersecting lines.

The front views of all lines of a plane which is perpendicular to V will all lie in one straight line.

The top and front views of lines in a profile plane are all in the same perpendicular to the ground line.

If one line of a plane is parallel to G. L., the plane is parallel to G. L. **A line is parallel to G. L. when both its top and front views are parallel to G. L.** This line with any point outside the line will represent a plane parallel to G. L.

Describe the position with respect to H, V, P, or G. L. of the plane ABC shown in the figures on page 57.

49. Lines on the plane. The two intersecting lines MN and OP, Fig. 33, represent a plane. The line AB, joining the point A in MN with the point B in OP, is another line of the plane. *ab* is its top and *a′ b′* its front view. The top view *a b* of the line can be drawn at random and the front view found by projecting the intersections with *m n* and *o p* to the front views *m′n′* and *o′p′*. These points determine the front view of the line. Had the front view of the line been drawn first, the top view could have been found in a similar manner.

A line which lies in a given plane and is parallel to H, is called a horizontal line of that plane. In Fig. 34, SS is such a line which lies in the plane of the lines MN and OP. **Its front view *s′ s′* is drawn first** and then its top view *ss* is found by locating the top views of the points X and Y where it crosses the given lines MN and OP. **A line which lies in a given plane and is parallel to V is called a frontal line of that plane.** FF, Fig. 35, is such a line which lies in the plane of MN and

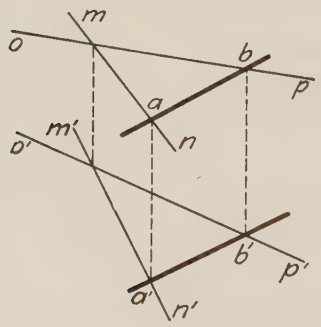
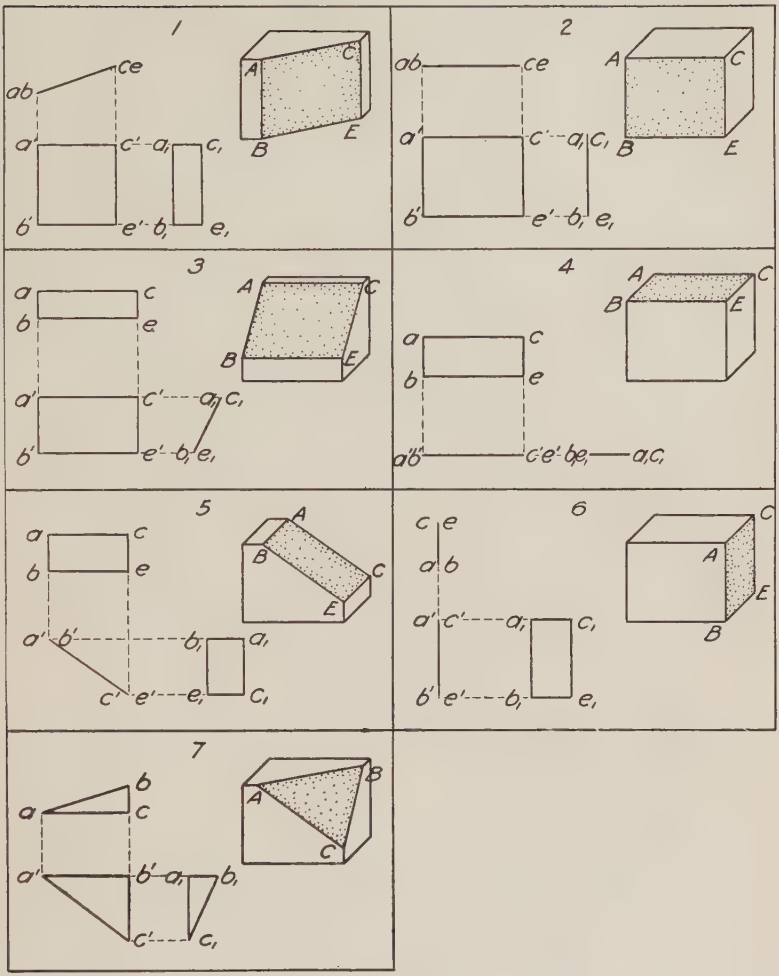


FIG. 33.—*Line AB on plane of MN and OP.*



Plane ABC in various positions with respect to H, V, P.

OP. **Its top view** *ff* **is drawn first** and then its front view *f' f'* is found. These lines which lie in a plane and are parallel to H or V, play an important part in the solution of many problems.

50. **Points on planes.** A point of a plane can be represented by placing the views of the point on the views of some line of the plane. If the top view of the point P which lies on the plane is given, the front view of P can be found as follows: Draw the top view of any line through the top view of P. Find the front view of the line by finding the front view of the points where it crosses the lines representing the plane. Then the front view of P is on the front view of this line. If the front view of P had been given, the top view could have been found in a similar manner.

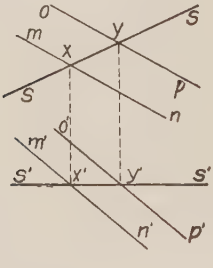


FIG. 34.—*SS horizontal line of plane MNOP.*

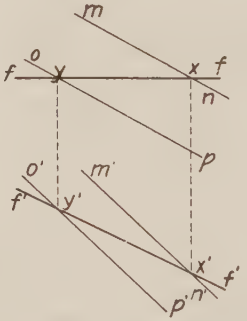


FIG. 35.—*FF frontal line of plane MNOP.*

All points of the plane which are a given distance, say 1″ higher or lower than any point P, lie on a horizontal line of the plane. The true distance which the point is higher or lower than P is shown in the front view. Likewise, all points of the plane which are a given distance farther back or farther toward the front than a point P lie on a frontal line of the plane. The true distance which the point is farther to the front or back than P is shown in the top view.

51. Problems.

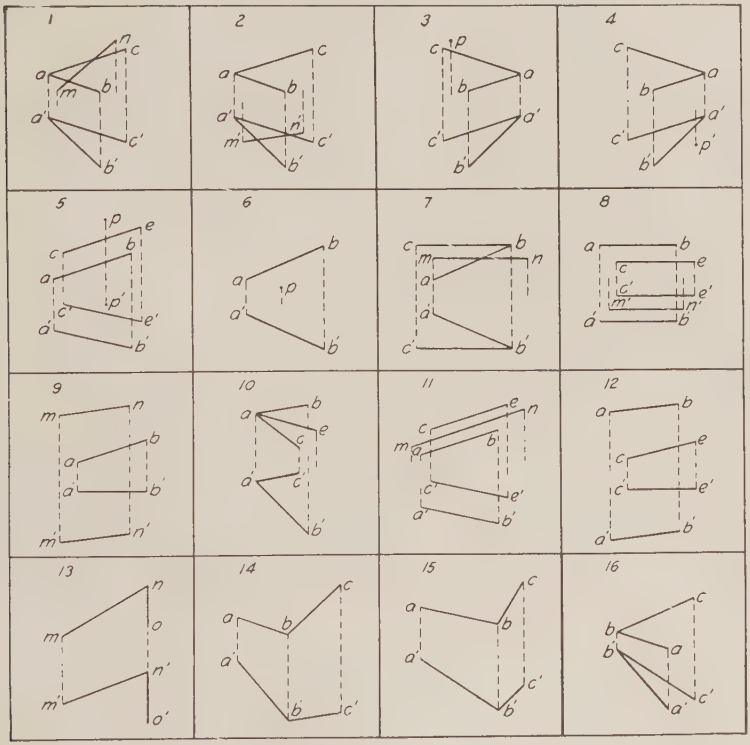
1. Given the top view of a line M (—1¼, ¼, ?) N (½, 1¾, ?) which lies on the plane B (0, ¼, —2) A (—1½, ¾, —½) C (¾, 1½, —1¼). Draw the front view of MN.

2. Given the front view of a line M (—1¼, ?, —1¼) N (½, ?, —1) which lies on the plane B (0, ¼, —2) A (—1½, ¾, —½) C (¾, 1½, —1¼). Draw the top view of MN.

3. Given the top view of a point P (—½, 1¾, ?) which lies on the plane B (0, ¼, —2) A (1½, ¾, —½) C (—¾, 1½, —1¼). Find the front view of P.

4. Given the front view of a point P (1¼, ?, —1¼) which lies on the plane B (0, ¼, —2) A (1½, ¾, —½) C (—¾, 1½, —1¼). Find the top view of P.

5. A plane is represented by two parallel lines A (—1¾, ¼, —1¼) B (½, 1, —1¾) and C (—1½, 1, —½) E (¾, 1¾, —1). Determine whether or not the point P (—¼, 2, —½) lies in the plane ABCE.



6. Represent a plane which contains the line A (—1¾, ¼, —¾) B (½, 1¼, —1¾) and is parallel to the ground line. P (—¾, 0, ?) is a point on this plane. Find the front view of P.

(If one line of a plane is parallel to G. L. the plane is parallel to G. L.)

7. Given the top view of a line M (—1¾, ⅞, ?) N (1, ⅞, ?) which lies on the plane A (—1¾, ¼, —¾) B (½, 1¼, —1¾) C (—2¼, 1¼, —1¾). Draw the front view of MN.

8. A plane parallel to the ground line is represented by two parallel lines A (—1½, 1. —1¼) B (¾, 1, —1¼) and C (—1, ½, —½) E (1¼, ½, —½). If the line M (—1¼, ?, —⅞) N (1, ?, —⅞) lies in the plane of AB and CE, find its top view.

9. Determine graphically whether or not the lines M (—1¼, 1½, —2¼) N (¾, 1¾, —2) and A (—¾, ⅛, —¾) B (1¼, ¾, —¾) intersect. Do not prolong the lines.

10. A plane is represented by two intersecting lines A (—1, 1½, —½) B (½, 1¾, —2) and AC (¼, ½, —¼). The top view of a line AE (¾, 1, ?) which passes through their point of intersection and lies in the plane of AB and AC is given. Find the front view of the line AE.

11. A plane is represented by two parallel lines A (—1¾, ¼, —1¼) B (½, 1, —1¾) and C (—1½, 1, —½) E (¾, 1¾, —1). The top view of another line M (—2, ½, ?) N (1¼, 1⅝, ?) of this plane is parallel to AB and CE. Find the front view of the line MN.

12. Determine graphically whether or not the lines A (—1¼, 1½, —2¼) B (¾, 1¾, —2) and C (—¾, ⅛, —¾) E (1¼, ⅝, —¾) intersect. Do not prolong the lines AB and CE.

13. Draw the top and front views of a horizontal line and a frontal line on the plane M (—1½, ¼, —1¾) N (1, 1¾, —¾) O (1, ½, —2¼).

14. Find the top and front views of the locus of all points which lie on a given oblique plane A (—1½, ¾, —½) B (0, ¼, —2¼) C (1½, 1¾, —2) and are 1″ lower than A.

15. Find the top and front views of the locus of all points which lie on a given oblique plane A (—1½, ¾, —¾) B (¾, ¼, —2¼) C (1½, 1½, —1½) and are 1″ farther back than B.

16. Find the top and front views of a point which lies on a given oblique plane A (0, —½, —2¼) B (—1½, 0, —½) C (¾, 1, —2) and is ¾″ higher than A and 1″ farther back than B.

52. A point and two lines are given by their top and front views. Represent a plane which contains the given point and is parallel to the given lines.

Let P be the given point and AB and MN the given lines.

Analysis. Through the given point P draw a line parallel to each of the given lines AB and MN. The plane of these two lines is the required plane.

Construction. Since parallel lines show as parallel lines in all views, draw through p lines parallel to $a\ b$ and $m\ n$ and through p' lines parallel to $a'b'$ and $m'n'$. These lines represent a plane which contains P and is parallel to the lines AB and MN.

In case AB or MN is a line of profile, use an end view to draw a line through P parallel to the line of profile.

To represent a plane which contains AB and is parallel to MN, draw a line CE which cuts AB and is parallel to MN. Then ABCE is the required plane.

To represent a plane which is parallel to a given plane, draw two intersecting lines parallel respectively to two lines on the given plane. The two intersecting lines represent the required plane.

53. Problems.

1. Represent a plane which contains the line A (—1¼, ⅞, —1) B (¾, 1¾, —1½) and is parallel to the line C (—1, 1½, —1¾) E (1, ½, —1⅛).

2. Represent a plane which contains a given point and is parallel to G. L.

3. Given two lines A (—1½, 1, —1¼) B (¾, 1, —1¼) and C (—1, ½, —½) E (1¼, ½, —½) which are parallel to G. L. Represent a plane which contains the point P (¼, 1¼, —1) and is parallel to the plane of the given lines.

4. Given a line of profile A (½, 2, —2) B (½, ½, —¾) and an oblique line C (—1¼, ¼, —1) E (¾, 1½, —1½). Represent a plane which contains the point P (—1¼, 1¼, —1¾) and is parallel to the given lines.

5. Represent a plane which contains the point P (—½, ¼, —1¾) and passes at equal distances from two other given points A (—1, 1¾, —½) and B (1, ¾, —1). (There are any number of solutions.)

6. Represents a plane which contains the line M (—1¼, ¼, —1½) N (½, ½, —¼) and passes at equal distances from two given points A (—1, 1½, —½) and B (1½, ½, —1¼). (Two solutions).

7. Represent a plane which contains the point P (—¾, ¾, —1¼) and passes at equal distances from three points A (—1¼, 1¾, —½) B (0, ½, —⅜) and C (1¼, 1¼, —2¼). (Four solutions).

8. Draw the top and front views of a plane which will cut a parallelogram from the tetrahedron A (—1, ¼, —2¼) B (—⅜, 2, —2¼) C (1¼, 1¼, —2¼) E (¼, 1¼, 0). The parallelogram need not be drawn.

9. Through the points A (½, ½, —1), B (—½, 1, —½), and C (0, 1½, —1½), draw three planes which are parallel and equidistant.

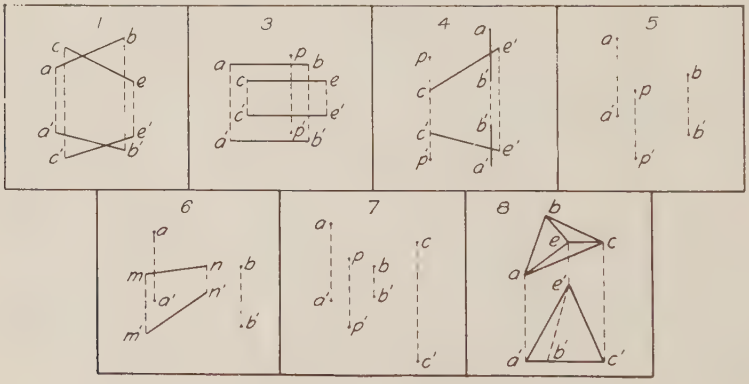
10. Through the points A (½, ½, —1), B (—½, 1, —½) and C (0, 1½, —1½), draw three planes which are parallel and equidistant, and such that one of the planes passes through P (—1, ¾, —¼).

11. Through the points A (0, 1, 0), B (1, ½, —1½), C (—½, 0, —1½), and E (—¾, 1¼, —1), draw four planes which are parallel and equidistant.

12. Draw, respectively, through the points A (½, ½, —1) and B (—½, 1, —½) and the line C (0, 1½, —1½) E (¼, 0, —½), three planes which are parallel and equidistant.

13. Represent a plane which is equidistant from a point A (¾, 1, —½) and a line B (—1, 1½, —½) C (¼, 0, —1½) and is parallel to the line E (1, 1½, —½) F (—½, 0, —1½).

14. Represent a plane which is equidistant from the points A (¾, 1, —½), B (—1, 1½, —½), and C (¼, 0, —1½) and is parallel to the line E (1, 1½, —½) F (—½, 0, —1½). (Three solutions).



INTERSECTIONS OF LINES WITH PLANES

FIRST METHOD—AUXILIARY VIEW

54. To find the point in which a line pierces a plane. Let the points A, B, and C, Fig. 36, represent the plane and EG the given oblique line.

Analysis. An auxiliary view taken by looking parallel to the plane will show the plane as one line and the oblique line as another line. Where these lines cross will be the auxiliary view of the point where the oblique line pierces the given plane. The top and front views of this piercing point can be found from the auxiliary view.

Construction. Draw the front view s' s' and then the top view ss of a horizontal line on the plane ABC. Picture the plane SSB over its top view. In the auxiliary view

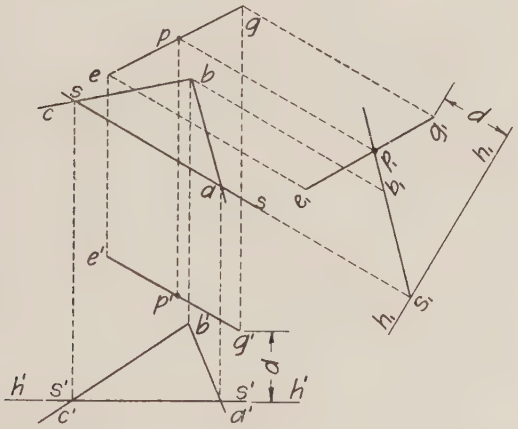


FIG. 36.—*Line EG pierces plane ABC at P.*

taken by looking in the direction of SS, h_1 h_1, at right angles to ss represents a horizontal plane at the level SS. Using this for a reference line, the lines b_1 s_1 and e_1 g_1 which represent respectively the plane ABC and the line EG, are determined. p_1, the intersection of b_1 s_1 and e_1 g_1,

is the auxiliary view of the point where EG pierces the plane ABC. Since P is on the line EG, its top view is p and its front view p'.

The same piercing point could have been found by taking an auxiliary view looking in the direction of a frontal line of the plane ABC.

55. Problems.

1. A (—2½, 1¼, —2⅝) B (—1¾, 0, —2⅝) C (—¾, ½, —2⅝) is the base of an oblique prism which has A (—2½, 1¼, —2⅝) E (—1¼, 2½, —1) for a lateral edge. Find the line of intersection of the prism with the plane M (—2¼, ⅜, —1) N (—1¾, 1¾, —2) O (—¼, 1¾, —2¼).

2. M (—2¼, 1½, —2½) N (—1, 2¼, —2½) O (—¼, 1, —2½) P (—1½, ¼, —2½) is the base of a right prism whose edges are 1½″ long. Find the line of intersection of the prism with the plane A (0, 2, —1) B (—1¼, ¾, —2½) C (—2½, 1¾, —2).

3. M (—2¼, 1½, —2½) N (—1, 2¼, —2½) O (—¼, 1, —2½) P (—1½, ¼, —2½) is the base of a regular pyramid with altitude 2″. Find the line of intersection of the pyramid with the plane A (—2½, 1¾, —1¾) B (0, 2, —1) C (—1½, 0, —2¼).

4. M (—2¼, ¾, —2½) N (—1½, 1¾, —2½) O (—½, ¾, —2½) is the base of an oblique pyramid with vertex at V (0, 2⅜, —½). Find the line of intersection of the pyramid with the plane A (0, 2, —2) B (—1¼, 0, —½) C (—2½, 1¾, —1½).

SECOND METHOD OF FINDING THE POINT IN WHICH A LINE PIERCES A PLANE

56. To find the points in which a given oblique line pierces the faces of a prism.

Let the oblique line and prism be given as shown in Fig. 37.

Analysis. mn may be considered the top view of a plane which contains the given line MN in space and is perpendicular to H. This plane cuts the triangle XYZ from the surface of the prism. The points where the sides of the triangle XYZ intersect the line MN will be the points where the line MN pierces the faces of the prism.

Construction. The top views of the points in which the edges of the prism pierce the plane which contains MN and is perpendicular to H are x, y, and z. The front views x', y', and z' of these points are located on the front views

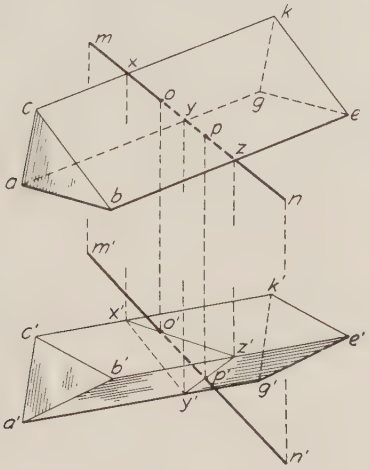


FIG. 37.—*Line MN pierces faces of prism at O and P.*

of the respective edges of the prism. By joining the points x', y', and z' with straight lines, the front views o' and p' of the piercing points are located. o and p are the top views of these points. The part of the line MN which is hidden by the prism is represented by dashed lines.

The same piercing points O and P could have been found by considering a plane which contains MN and is perpendicular to V. m′ n′ represents this plane in the front view. This plane cuts a triangle (not shown in Fig. 37) from the prism which crosses MN in the piercing points O and P.

57. To find the point in which a given oblique line pierces any plane.

Note. This is the same problem as that taken up in the preceding article. It is here stated in more abstract form, considering the surface pierced by the line as a plane of indefinite size rather than the surface of a solid.

Let AB and AC, Fig. 38, be the lines which represent the plane and MN the given oblique line.

Analysis. Picture a plane which contains the given line and is perpendicular to H. The intersection of this plane with the given plane crosses the given line at the required point.

Construction. *mn* is the top view of a plane which contains MN and is perpendicular to H. AB pierces this plane at Y and AC at X. *x'y'*, the front view of the inter-

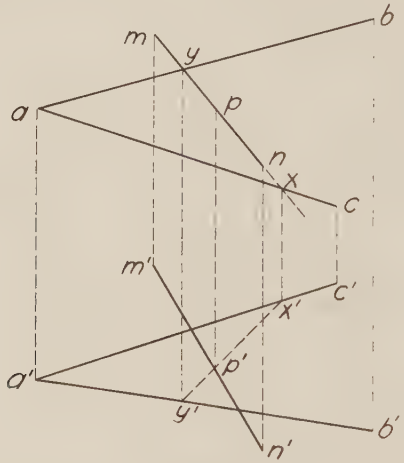


FIG. 38.—*MN pierces plane of AB and AC at P.*

section of the two planes, crosses *m'n'* at *p'*, the front view of the required point; *p* is the top view of this point.

The point in which the given line pierces the plane ABC can be found also by means of a plane which contains MN and is perpendicular to V. The line of intersection of this plane with the plane ABC cuts MN at the piercing point.

58. Problems.

1. Find the point where the line E (—1¼, ½, —1) G (1¼, 1¾, —2¼) pierces the plane A (1¼, 1, —¾) B (—1¼, 1¾, —1¾) C (¼, ¼, —2).

2. Find the point in which the line A (—1½, 1¼, —⅝) B (1½, ½, —2⅛) pierces the plane C (—1, ¾, —1¼) E (1, ¼, —½) G (—1, 1¾, —2¼) K (1, 1¼, —1½).

3. Find the point in which the line C (—1, 1⅛, —2¼) E (1½, 1⅛, —¾) pierces the plane A (⅝, 1½, —1¾) B (—1¼, 1½, —1¾) G (1¼, ¼, —¼).

4. Find the point in which the line A (—1, 1¾, —¾) B (1, ¼, —2¼) pierces the plane C (—½, 1½, —2) E (—2½, 1½, —2) G (0, ¼, —½).

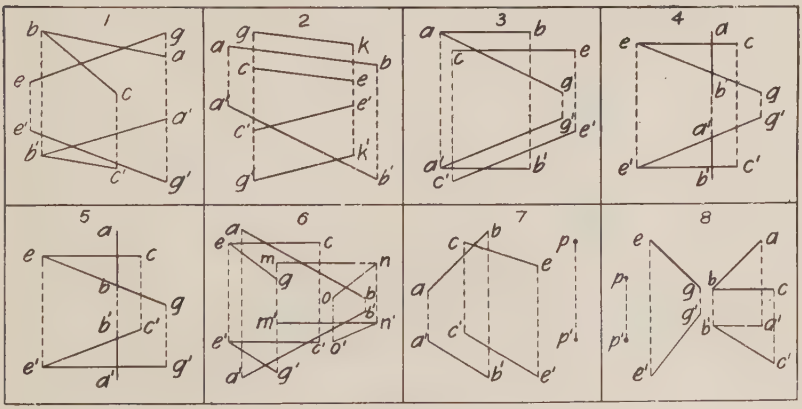
5. Find the point in which the line A (—1, 2, —2¼) B (—1, ¼, —½) pierces the plane C (—½, 1½, —½) E (—2½, 1½, —2) G (0, 0, —2).

6. Find the length of the part of the line A (—1½, 3⅜, —4¼) B (2, ⅞, —1¾) which is included between the planes C (½, 3, —3) E (—2½, 3, —3) G (—¼, 1⅜, —4⅛) and M (—¼, 2, —2) N (2½, 2, —2) O (1, 1, —3). (Use a 7½″ x 11″ rectangle for this problem.)

7. Draw the top and front views of a line which passes through P (1½, 1½, —1¼) and touches the lines A (—1½, ¼, —¾) B (—¼, 1¾, —2) and C (—¾, 1⅝, —½) E (¾, ⅞, —2).

8. Draw the top and front views of a line which passes through P (—2½, ½, —1), is parallel to the plane A (2½, 1¼, —¾) B (½, ¼, —¾) C (2¾, ¼, —1⅞), and touches the line E (—1½, 1½, —2¼) G (0, ¼, —½).

9. Draw the top and front views of a line which touches the three lines: A (—½, ½, —½) B (½, 1½, —1½), C (—1, 1, 0) E (0, ½, —1½), and F (0, 1½, 0) G (1, ½, —1).



10. A drill hole is driven at P (—250′, 100′, —100′) N 45° E dipping 60°. Find the length of the drill hole to the point where it strikes the wall of a vein of ore whose strike is S (—250′, 300′, —300′) S (50′, 150′, —300′) and whose dip is 45° to the Southwest. What is the difference in elevation between P and the point where the drill hole strikes the vein? (Scale 1″=100′. Use a 7½″ x 11″ rectangle for this problem).

11. Find the points in which the line A (—2, 3½, —3⅞) B (1, ½, —2¾) pierces the tetrahedron whose vertex is V (½, 2½, —1½) and whose base is C (—1½, 2½, —4) E (—1½, ½, —4) G (½, ½, —4). (Use a 7½″ x 11″ rectangle for this problem).

59. Two planes are each represented by two of their lines. Find the line of intersection of the planes.

Let AB and BC, Fig. 39, be the lines of one plane and MN and OP the lines of the other plane.

Analysis. Find the points where two lines of one plane pierce the other plane. The line joining these piercing points is the required line of intersection of the planes.

Construction. MN pierces the plane of AB and BC at the point K. OP pierces the plane of AB and BC at the point G. Then *kg* and *k′ g′* are the top and front views respectively of the required line of intersection of the planes.

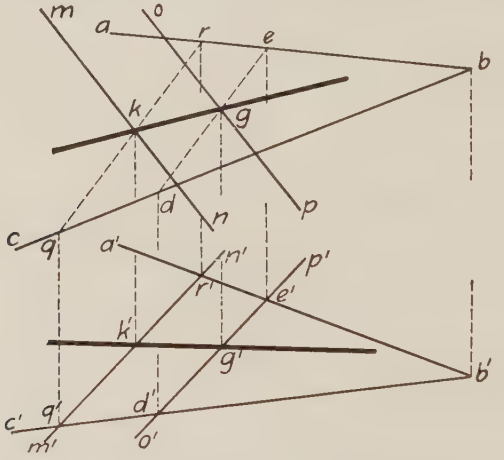


FIG. 39.—*KG line of intersection of planes ABC and MNOP.*

Sometimes the lines which represent one plane do not cross the lines which represent the other plane within the limits of the drawing, or they cross in such a manner that the above method cannot be used to find the intersection

of the planes. In this case points of the line of intersection can be found by drawing additional lines of one plane which do cross the lines of the other plane within the limits of the drawing. The piercing points of these additional lines determine the line of intersection of the planes.

60. Problems.

Use a 7½" x 11" rectangle for each of the following problems:

1. Draw the top and front views of the line of intersection of the planes A (—2, 1, —3) B (½, 3½, —1¾) C (—1, ¼, —3⅝) E (1½, 2¾, —2⅜) and M (—1, 2¾, —1) N (1¼, 1⅝, —3¾) O (—1¾, 2⅛, —1¾) P (½, 1, —4½).

2. Draw the top and front views of the line of intersection of the planes A (—2½, ½, —3) B (0, 3, —⅞) C (1⅞, 1¾, —4⅛) and M (—2½, 2½, —2½) N (2, 2½, —2½) O (—2½, 1½, —1½) P (2½, 1½, —1½).

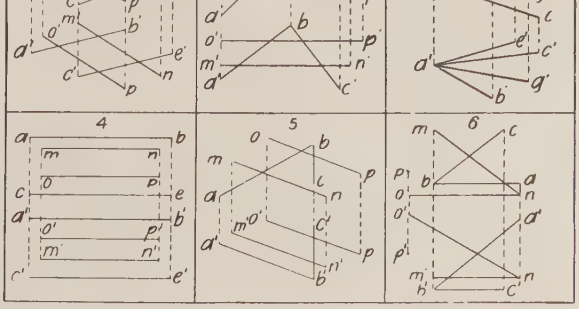
3. Draw the top and front views of the line of intersection of the planes B (½, 4, —4) A (—2, 1½, —2¾) C (2, ¼, —2¼) and E (1, 2½, —1¾) A (—2, 1½, —2¾) G (1½, 1, —3½).

4. Draw the top and front views of the line of intersection of the planes A (—2, 2½, —1¾) B (2, 2½, —1¾) C (—2, ½, —4) E (2, ½, —4) and M (—1¾, 2, —3¼) N (1¾, 2, —3¼) O (—1¾, 1¼, —2⅜) P (1¾, 1¼, —2⅜).

5. Draw the top and front views of the line of intersection of the planes A (—2½, 1½, —2½) B (1, 3¼, —3¾) C (1, 1¾, —1⅝) and M (—2, 2½, —2) N (1½, 1½, —3¼) O (—1, 3½, —1⅜) P (2½, 2½, —2⅝).

6. Draw the top and front views of a line which passes through the point P (—2, 1½, —2¾) and is parallel to both planes A (2, 1, —1¼) B (—1, 1, —4¼) C (1½, 3½, —4¼) and M (—1, 3½, —3¾) N (2, ½, —3¾) O (—2, ½, —1⅛).

7. Given the tetrahedron A (—1½, 1, —1½) B (2, 1¾, —3½) C (—½, 3½, —3¾) E (1, ¾, —¼). Represent a plane which contains the point P (—⅜, 2, —2½) and cuts a parallelogram from the given tetrahedron. Draw the top and front views of the parallelogram.



THE ANGLE BETWEEN ANY TWO PLANES

A view of any two planes taken in the direction of their line of intersection will show each plane as a straight line. The angle between these lines is the true size of the angle between the given planes.

61. To find the angle which a given oblique plane makes with a horizontal plane.

Fig. 40 represents a triangular prism resting on one of its lateral faces.

Let the given oblique plane be the face ABCE of the prism, and let the face upon which the prism rests be the horizontal plane.

Analysis. A view taken by looking in the direction of the line of intersection of the planes will show each plane as a straight line. The angle between these lines is the angle between the planes.

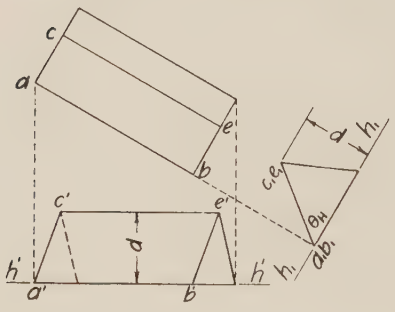


FIG. 40.—θ_H *angle plane ABCE makes with H.*

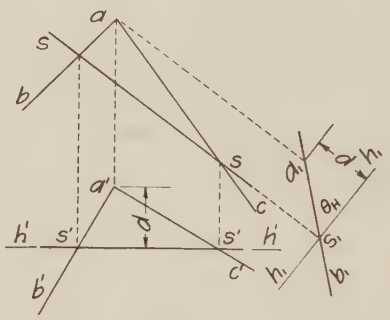


FIG. 41.—θ_H *angle plane ABC makes with H.*

Construction. ab is the top and $a'\ b'$ the front view of the lateral edge of the prism, which is the line of intersection of the given planes. In the auxiliary view, $h_1\ h_1$ which is at right angles to ab, represents the horizontal

plane upon which the prism rests. h' h' is the front view of this plane. The distance d which the edge CE is above the plane of the opposite face is taken from the front view and used to locate c_1 e_1 in the auxiliary view. a_1 c_1 represents the plane of the face of the prism in the auxiliary view and θ_H is the angle which this face makes with the horizontal plane upon which the prism rests.

62. To find the angle which a given oblique plane makes with a horizontal plane when the oblique plane is represented by two intersecting lines AB and AC. Fig. 41.

In the front view the horizontal plane is represented by one line h' h' parallel to G. L. The top view of this horizontal plane is unlimited in extent, covering all of H, and therefore need not be represented by any straight lines. The intersection of this horizontal plane with the plane ABC is the horizontal line SS. Picture the plane SSA over its top view, SS coinciding with ss and A the distance d above a. In the auxiliary view taken by looking in the direction of the line SS, h_1 h_1, which is at right angles to ss, represents the horizontal plane. The distance d taken from the front view, locates a_1 in the auxiliary view. a_1 b_1 represents the oblique plane ABC in the auxiliary view and θ_H the angle which it makes with the horizontal plane.

63. Conversely, to represent a plane which makes a given angle with H.

There are any number of planes which make a given angle, say 30°, with H. Any horizontal line SS, Fig. 42, can be drawn and a plane represented which contains this line and makes 30° with H. In the auxiliary view taken by looking in the direction of SS, h_1 h_1 at right angles to ss, represents H and s_1 a_1, at 30° with h_1 h_1, represents the required plane. The top view of the point A can be considered to be any point on the line ab, parallel to ss. The

front view of AB is located by taking the distance d from the auxiliary view and setting it above h' h' in the front view, a' b' being parallel to s' s'. The required plane is represented by the two parallel lines SS and AB or by SS and any point on the line AB. Other oblique lines of the plane can be drawn by joining points on AB with points on SS.

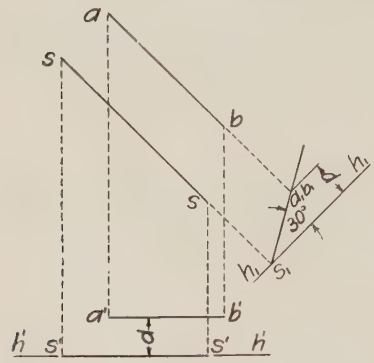


FIG. 42.—*Plane SSAB 30° with H.*

There are two planes which contain SS and make 30° with H. The other one would be represented in the auxiliary view by a line passing through s_1, sloping downward and making 30° with h_1 h_1.

64. To find the angle which a given oblique plane makes with a plane parallel with V.

Let ABC, Fig. 43, represent the oblique plane and vv the top view of the plane which is parallel to V.

Analysis. A view taken by looking in the direction of the line of intersection of the planes will show each plane as a straight line. The angle between these lines is the angle between the planes.

Construction. ff is the top view and $f'f'$ the front view of the line of intersection of the planes. Picture the plane FFA over the front view, FF coinciding with $f'f'$ and A the distance d behind its front view a'. In the auxiliary view, v_1 v_1 which is at right angles to $f'f'$, represents the plane which is parallel to V. The distance d which the point A is back of this plane is taken from the top view and used to locate a_1 in the auxiliary view. a_1f_1 represents the oblique plane in the auxiliary view and θ_v is the angle

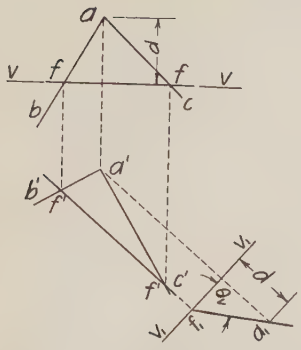


FIG. 43.—*θ_v angle plane ABC makes with V.*

which this plane makes with the plane parallel with V. The oblique plane also makes this same angle θ_v with V.

In the auxiliary view, a_1 is placed nearer the front view than the line $v_1\ v_1$ when the point A is in front of the vertical plane as determined by the top view. If A is back of the vertical plane then a_1 is placed on the other side of $v_1\ v_1$.

65. Conversely, to represent a plane which makes a given angle with V.

There are any number of planes which make a given angle, say 45°, with V. Any frontal line FF, Fig. 44, can be drawn and a plane represented which contains this line and makes 45° with V. In the auxiliary view taken by looking in the direction of FF, $v_1\ v_1$ at right angles to $f'\ f'$, represents V and $f_1 a_1$, 45° with $v_1\ v_1$, represents the required plane. Each point of the line $f_1\ a_1$ represents a line which is parallel to FF and lies in the required plane. a_1 b_1 represents such a line. a' b' parallel to $f'f'$ is its front view. The top view of AB is located by taking the distance d from the auxiliary view and placing it back of vv in the top view, ab being parallel to ff. The required plane is represented by the two parallel lines FF and AB, or by FF and any point on the line AB. Other oblique lines of

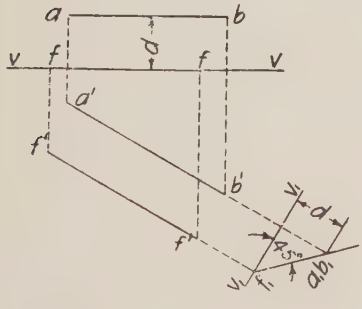


FIG. 44.—*Plane FFAB 45° with V.*

the plane can be drawn by joining points on AB with points on FF.

There are two planes which contain FF and make 45° with V. The other one slopes forward making 90° with the plane of FF and AB.

TERMS USED IN GEOLOGY AND MINING

66. A vein of ore, as the term is used in this textbook, has two supposedly parallel walls between which the vein material lies. These walls are indefinite in extent, the distance between the walls is the thickness of the vein.

A bed or stratum of rock. Rocks are frequently found in layers or beds. The plane of the bed may be horizontal or it may be tilted at an angle with the horizontal.

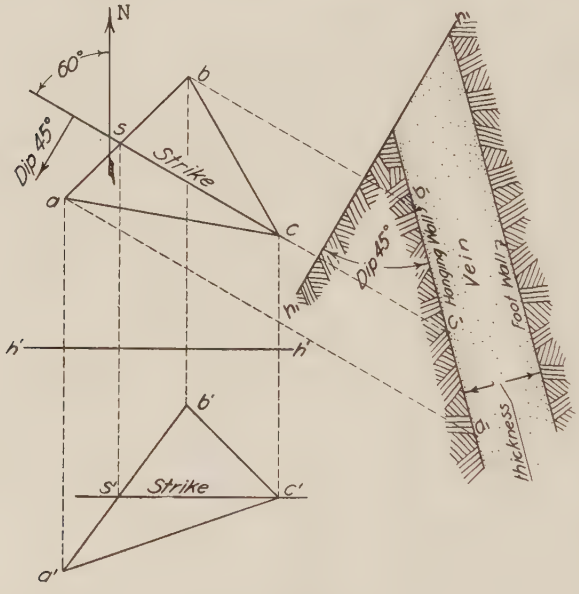


FIG. 45.—*Direction of strike, north 60° west. Dip of vein, 45° to south west.*

The direction of strike of a vein of ore or a bed of rock is the direction of a horizontal line on either wall of the vein of ore or on the bedding plane of a rock.

The dip or inclination of a vein of ore or bed of rock is the angle which the plane of the wall of the vein or bedding makes with the plane of the horizon and is always measured at right angles to the strike.

The direction of the strike is given in terms of its bearing from some point of the compass as the true north. As an example, N $\theta°$E means that the strike line runs $\theta°$ to the right of a perpendicular to the G. L., the top of the drawing being north. The dip or inclination of the vein is shown in an auxiliary view taken in the direction of the strike line, as shown in Fig. 45.

Drill holes are holes of small diameter, usually less than two inches, made for prospecting or informational purposes. The drill hole may be vertical or inclined. The direction of the drill hole is given by its **bearing,** which indicates the direction of its top view from some point of the compass as the north, and its **dip** which gives the angle between the drillhole and the plane of the horizon.

A fault is a fracture or break in the earth's crust along which displacement of one side with respect to the other has taken place.

A fault surface is the surface of a fracture along which displacement takes place, and if without notable curvature is called **a fault plane.** In the movement the fault surfaces may scratch and polish each other. These **scratches** or **striae** on the walls show the direction of the movement.

67. Problems.

1. Find the angle which the plane A (—2½, ¾, —2¼) B (—1, 1½, —¾) C (½, ¼, —1¾) makes with H.

2. A (—2¾, 1⅝, —4) B (—1⅝, 3⅛, —4) C (—⅛, 2, —4) E (—1¼, ½, —4) is the base of a regular square pyramid whose altitude is 2½". If V is the vertex, find the angle which the face VBC makes with the plane of the base. (Use a 7½" x 11" rectangle for this problem.)

3. Vertical drill holes are driven to a certain vein of ore at three points A (—550, 400, ?) B (—250, 700, ?) and C (50, 200, ?). The drill holes strike the vein at 650', 1050' and 750' above sea level, respectively. If the direction of the G. L. is considered to be East and West, find the strike and dip of the vein. Scale 1"=200'. (Use a 7½" x 11" rectangle for this problem.)

4. S (—¼, ¼, —1½) S (2⅜, 1¾, —1½) is a horizontal line of a plane which makes 30° with H. Represent the plane in the top and front views.

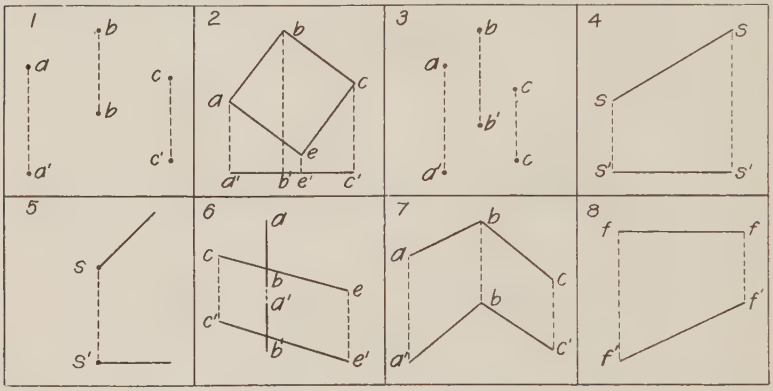
5. The strike of a certain ore vein is N 45° E from the point S (0, ¼, —½). Represent one wall of the vein when the dip is 45° to the NW.

6. Find the angle which the plane of A (1, 3½, —2) B (1, 1¼, —3½) and C (0, 2½, —2½) E (2¾, 1⅛, —3⅞) makes with H and V. (Use a 7½″ x 11″ rectangle for this problem.)

7. Find the angle which the plane A (—2¾, ¾, —2¼) B (—1¼, 1½, —¾) C (¼, ¼, —1¾) makes with V.

8. Represent a plane which contains the line F (—2⅜, 1, —2¼) F (¼, 1, —¾) and makes 45° with V.

9. Through a given point P draw three lines whose top and front views are respectively parallel. Do these lines lie in one plane? If the three lines lie in one plane, find the angles which the plane makes with H and V.



10. Through a given point P draw three lines whose top and front views make respectively the same angle with the ground line. Do these lines lie in one plane? If the three lines lie in one plane, find the angles which the plane makes with H and V.

11. Draw the top and front views of a line which passes through the point P (0, 1, —1), is parallel to the plane A (—¼, 1⅛, —1¾) B (1, 1½, —1¼) C (1, ⅜, —⅝), and has its top and front views parallel to each other.

12. A stratum of rock whose strike is N 60° W outcrops on the vertical wall of a railway cut in lines which dip 45° with the horizontal. If the railway cut bears due north, what is the true dip of the stratum of rock? (G. L. runs east and west.)

13. A stratum of rock whose strike bears N 60° E outcrops on the vertical wall of an east and west railway cut in lines which make 45° with H. A second railway cut runs N 15°E. What angle will the lines of outcrop of the stratum make with H on the vertical wall of this second cut? What is the true dip of this stratum of rock? (G. L. runs east and west.)

68. Two planes are each represented by the top and front views of two of their lines, find the angle between the planes.

Let ABC and EFG, Fig. 46, be the given planes.

Analysis. If an auxiliary view of the two planes be taken by looking in the direction of their line of intersection, each plane appears as a line and the angle between the lines is the true size of the angle between the planes.

Construction. The line of intersection MN of the two given planes ABC and EFG is found by the method of Art. 59. Picture the line MN and the points A and E over the top view and take the first auxiliary view to show the intersection MN true length. The distances d and d_1 are taken

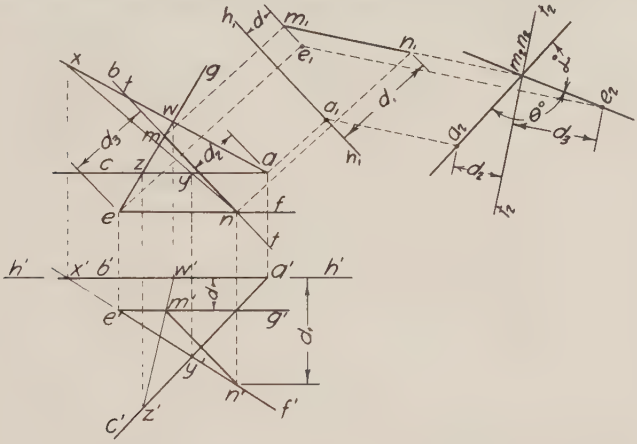


FIG. 46.— α° *or* Θ° *angle between given planes ABC and EFG.*

from the front view and used to locate m_1 n_1. In this view, e_1 represents a point on one of the given planes and a_1 a point on the other. Picture the line MN and the points A and E over the first auxiliary view and take the second auxiliary view in the direction of the line MN. In this view, MN appears as a point and consequently the given planes as lines. The distances d_2 and d_3 are taken from the top view and used to locate a_2 and e_2 in the second auxiliary view. This view shows the true size of the angles between the planes to be α° and θ°; the angles are supplementary.

69. Problems.

1. Find the angle between the planes A (—2, ¾, —¾) B (—½, ¾, —1⅝) C (—½, 2¼, —¾) and E (—½, ¼, —½) F (—1¾, ¼, —1¼) G (—1¾, 2½, —½).

2. Find the angle between the planes A (—2½, ½, —2) B (—1½, 2, —½) C (—¾, 1, —1) and M (—1½, ¼, —2) N (—¾, 1½, —¼) O (—2½, 2, —1½).

3. Find the angle between the planes A (—2½, 0, —2½) B (—1, 1¾, —1) C (—1½, —½, —2¼) E (0, 1¼, —¾) and M (—2¼, 1¼, —1) N (—1, 0, —2½) O (—1½, 2, —1) P (—¼, ¾, —2½).

4. The direction of strike of a stratum of rock is N 30° E from the point O (0, ½, ?) and the dip of the stratum is 45° N. W. The strike of a fault plane is N 45° W from the point O, and the dip of the fault plane is 60° N. E. Find the true size of the angle between the stratum of rock and the fault plane. (It is not necessary to draw the front view.)

70. To represent a plane which contains a given line and makes a given angle with a given plane.

As an example let it be required to represent a plane which contains the line AB, Fig. 47, and makes 60° with H.

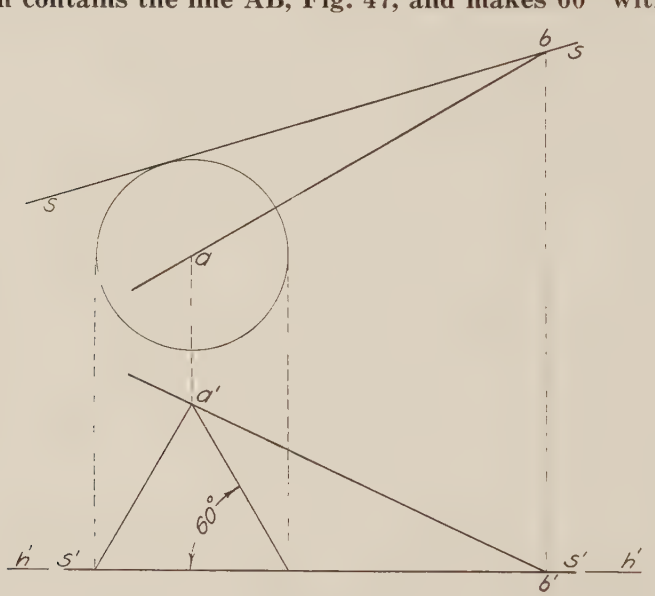


FIG. 47.—*Plane ABSS contains the given line AB and makes 60° with H.*

Analysis. A plane to contain a given point P and make a given angle, say $\theta°$ with a plane T, must be tangent to a right circular cone with vertex at P, base in the plane T,

and elements $\theta°$ with plane T. There are any number of planes which satisfy these conditions. There are usually two planes which contain a given line AB and make $\theta°$ with plane T. These planes contain the line AB and a line tangent to the base of a right circular cone, through the point where AB pierces the plane T. This cone has its vertex on the line AB, its base on the plane T, and elements $\theta°$ with T. If the line AB pierces the plane T outside the circle of the base of the cone, there are two solutions to the problem; if AB makes the same angle with T as the elements of the cone, there is only one solution; if AB pierces the plane T inside the circle of the base of the cone, the problem is impossible of solution.

Construction. Take any point of the line AB, as A, for the vertex of a cone the elements of which make 60° with H. The horizontal plane which has $h'h'$ for a front view can be placed at any elevation. This plane cuts a circle from the cone and a point B from the line AB. From b in the top view, draw $s\ s$ tangent to the circle which represents the base of the cone. SS is a horizontal line of the required tangent plane, $s's'$ is its front view. The lines SS and AB determine the required tangent plane. Another tangent to the base of the cone from B can be drawn which with AB will be a second solution of the problem.

A plane containing AB and making 60° with V could have been constructed. In this case the base of the cone would have been in a vertical plane and would have shown as a circle in the front view.

To represent a plane which contains AB and makes 60° with any oblique plane T, requires two successive auxiliary views. The first auxiliary view shows plane T as a line and the second auxiliary view shows the base of the cone on T as a circle. The tangent plane is represented in the first and second auxiliary views and then the top and front views of the plane are drawn.

71. Problems.

1. Represent the top and front views of a plane which contains the line A (—1¼, ½, —1) B (1¼, 1¾, —2) and makes 45° with H.

2. Represent the top and front views of a plane which contains the line A (—1¼, ¾, —¾) B (1¼, 2, —2) and makes 60° with V.

3. Represent the top and front views of a plane which contains the line A (—2¾, ½, —¼) B (—1, 1¾, —1½) and makes 60° with the profile plane.

4. Find a point P on the line M(⅝, ½, —⅜) N(—¼, 1⅛, —1¼) which is equidistant from the planes A (—¾, ½, —½) B (¾, 1½, —½) C (¾, ½, —½) and A (—¾, ½, —½) B (¾, 1½, —½) E (¼, ½, —1½).

5. Locate the center and find the diameter of a sphere which is tangent to planes through the origin parallel to H and V, and has its center on the line A (—1, 1, 0) B (1, 0, —1½).

PLANE FIGURES

72. If two views of the magnitudes which represent a plane are given, the position of the plane is determined. These magnitudes may form some geometrical figure as an angle, a triangle, a square, etc., or such a geometrical figure may be on the plane in addition to the magnitudes which represent the plane. A view of such a plane figure taken with the line of sight perpendicular to the plane will show the figure in its true size and shape. Consequently, if the plane of the figure is parallel to H, V, or P, the top, front, or end view respectively will show the true size of the figure. If the plane of the figure is oblique to both H and V, the true size of the figure can be found by rotating the plane, and the figure also, until it is parallel to H, when the top view will show the true size, or else parallel to V, when the front view will show its true size.

In order to rotate the plane of a figure until it is parallel to some other plane, the axis of rotation must be parallel to the line of intersection of the two planes. If, for example, the plane of the figure is to be rotated until it is parallel to H, in order for the top view to show the true size of the figure, the axis must be a horizontal line on the plane of the figure or an axis parallel to this horizontal line.

If the figure is to be rotated until it is parallel to V, the axis must be a frontal line of the plane or a line parallel to it.

Another method for finding the true size of a figure when the plane of the figure is oblique to H and V, is to take two auxiliary views of it. The line of sight for the first auxiliary view must be parallel to the picture plane of one of the given views and also from such a direction that the plane of the figure appears as a line. The second auxiliary view is derived from the first and has the line of sight perpendicular to the plane of the figure, thus showing the true size of the figure. For an illustration of this method see Art. 79.

73. Two intersecting lines are represented by their top and front views. Find the true size of the angle between them.

Let AO and BO, Fig. 48, be the given lines.

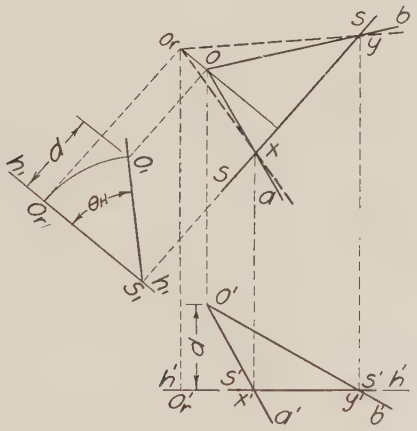


FIG. 48.—*x* o_r *y true size of angle AOB.*

Analysis. Draw a horizontal line in the plane of the given lines. Revolve the given lines about the horizontal line as an axis until the plane of the lines is parallel to H. The lines do not change their relative position during the

revolution. The top view of the angle in the revolved position shows the true size of the angle between the lines.

Construction. $s'\ s'$ is the front and ss the top view of a horizontal line on the plane of AO and BO. In the auxiliary view taken by looking in the direction of the axis, $h_1\ h_1$ represents a horizontal plane at the level of the axis, $h'\ h'$ is the front view of this plane. The distance d taken from the front view locates the point o_1 in the auxiliary view. The line $o_1\ s_1$ is the auxiliary view of the plane AOB and θ_H, the angle through which it must be revolved to bring it into a horizontal position. When the plane is in a horizontal position, the auxiliary view of O is at o_{r1}, its top view at o_r, and its front view at o'_r. The points X and Y where the lines OA and OB intersect the axis do not move during the revolution. Therefore xo_ry is the true size of the angle AOB.

The problem can be solved by revolving the plane of the angle until it is parallel to V about a frontal line as an axis. Then the front view shows the true size of the angle.

74. To bisect an angle. When the angle is shown in its true size, the bisector can be drawn. The bisector can then be revolved back until the plane of the angle is in its original position and the top and front views of the bisector found. In general, the views of the bisector will not bisect the views of the angle.

The view of an angle can be larger than, equal to, or smaller than the angle itself. **The view of a right angle is a right angle when one side of the angle is parallel to the plane upon which the view is taken.**

75. By the method explained in Art. 73, the true size of any plane figure can be found if the figure is represented by two of its views. If, for example, the top and front views of a triangle or quadrilateral are given, the true size of the figure can be found by revolving its corners about any horizontal axis which lies in the plane of the figure un-

til each corner reaches the level of the axis. Joining the corners thus located gives the true size of the figure in the top view.

76. Problems.

1. Find the true size of the angle A (—2½, ¾, —2¼) B (—1, 1½, —1) C (½, ¼, —2) by using a frontal line as an axis.

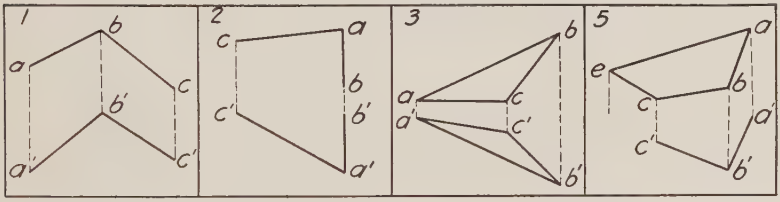
2. Draw the top and front views of the bisector of the angle B (—½, 0, —¾) A (—½, 1¾, —2¼) C (—2¾, 1, —½).

3. Draw the top and front views of the bisectors of all the angles of the triangle A (—2½, 0, —¾) B (0, 1⅜, —2⅛) C (—⅝, 0, —1).

4. Find the top and front views of the center of the circle whose circumference passes through the points A, B, and C given in problem 3.

5. Draw the top and front views of the quadrilateral A (½, 1¾, —2⅞) B (0, ¼, —4) C (—1½, 0, —3⅜) E (—2¾, ⅝, ?) and find its true size and shape. (Use a 7½" x 11" rectangle for this problem.)

6. Find the true size of the angle between two guy wires which are anchored at A (—65', 32.5', —72.5'), and B (10', —10' —90') and run to the top of a 50' vertical stack with base at C (—15', 57.5', —82.5'). What is the length of the wires? (Use a 7½" x 11" rectangle for this problem. Scale 1"=20'.)



77. A point and a line are represented by their top and front views. Draw the top and front views of a line which passes through the given point, makes a given angle with the given line, and cuts the given line.

Let P be the given point, 60° the given angle, and AB the given line, Fig. 49.

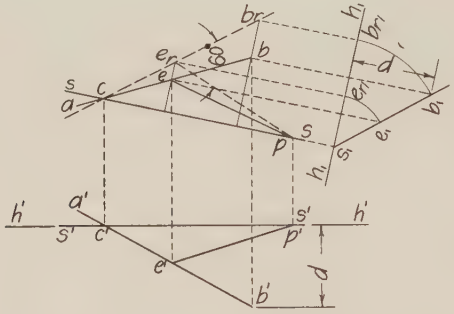


FIG. 49.—*Line PE 60° with given line AB.*

Analysis. Revolve the plane of the point and line about a horizontal axis until it is parallel to H. From the revolved position of the point, draw a line which makes the angle 60° with the revolved position of the given line. When this line is revolved back with the plane to the original position of the plane, it will be the required line.

Construction. s' s' is the front view and ss the top view of a horizontal line which lies on the plane of P and AB and cuts AB at the point C. When the plane is revolved about SS as an axis, B moves to B_r while P and C remain fixed, being on the axis. $b_r c$ is the top view of the revolved position of AB. Through p draw pe_r, making the required angle 60° with $b_r c$. e and e' are the top and front views of the point E after the counter revolution of the plane. Then pe is the top and p' e' the front view of the required line.

78. To construct the top and front views of any figure lying on a given oblique plane.

For example, let it be required to construct a 2″ square lying on the plane ABC, Fig. 50, having one corner of the square at A and another corner ¾″ lower than A. A horizontal line SS is first drawn on the plane ABC, every point

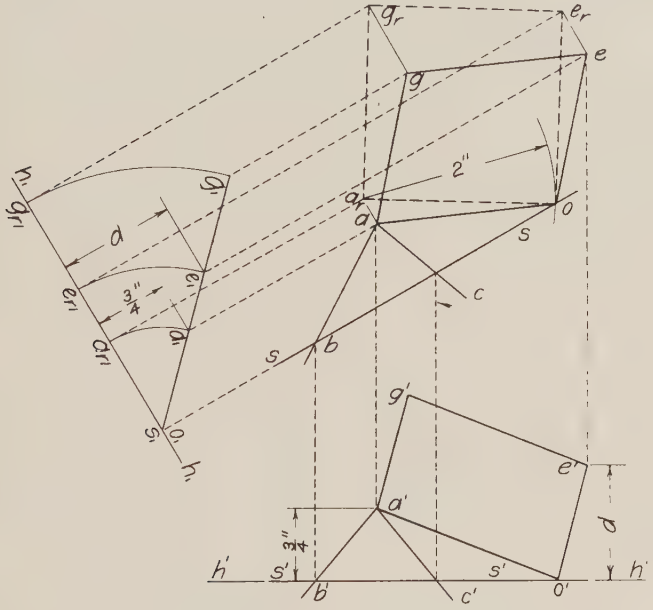


Fig. 50.—*Square AOEG lies on given plane ABC.*

of which is ¾″ lower than A; s' s' being found first and then ss. The plane ABC is revolved into a horizontal position about SS as an axis, a_r being the top view of A after revolution. With a_r as a center and a 2″ radius, strike an arc cutting ss at o. Construct a square with $a_r o$ as one side. Revolve this square into the plane ABC about SS as an axis and locate its top and front views. Then AOEG is the required square. The square could take several other positions on the plane and still satisfy the conditions of the problem.

79. To solve a plane figure problem by the two successive auxiliary view method.

To illustrate this method, let it be required to draw the top and front views of the bisectors of the angles of a given triangle.

Let ABC, Fig. 51, be the given triangle.

Analysis. The first auxiliary view is taken in the direction of a horizontal line on the plane of the triangle and shows this plane as a straight line. The line of sight for the second auxiliary view is perpendicular to the plane of the figure and this view shows the true size of the triangle. In this view the bisectors of the angles are drawn. From the second auxiliary view, the first auxiliary, the top, and front views of the bisectors are found.

Construction. In Fig. 51, draw the front and top views of the line SS. Picture the triangle ABC over the top view,

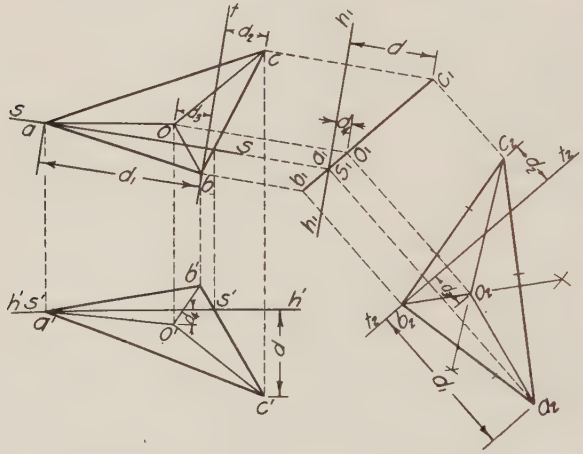


FIG. 51.—*OA, OB, OC bisectors of angles of triangle—auxiliary view method.*

A at *a*, B above *b*, and C the distance *d* below *c*. Let the plane T which contains the point B and is perpendicular to the line SS be the picture plane upon which the first auxiliary view is taken. After the triangle is projected upon the plane T, the plane is laid on the paper to the right

of the top view. In the first auxiliary view, the point B should be pictured at b_1, C the distance d_2 above c_1, and A the distance d_1 below a_1. In the second auxiliary view the plane T appears as the line $t_2\ t_2$, parallel to $b_1\ c_1$ and any convenient distance from it. a_2 is located by means of the distance d_1 and c_2 by means of d_2, both taken from the top view. $a_2\ b_2\ c_2$ is the true size of the triangle ABC. The bisectors of the angles of the triangle meet at o_2. From the second auxiliary view of O, the first auxiliary, the top, and front views are secured. *o a*, *o b*, and *o c* are the top and *o'a'*, *o'b'*, and *o'c'*, are the front views of the bisectors of the angles of the triangle ABC.

The method of solving plane figure problems by means of two successive auxiliary views usually gives more accurate construction than the rotation method and does not necessitate the drawing of more lines.

80. Problems.

1. Draw the top and front views of a line which represents the shortest distance from the point P (—½, 0, —1) to the line A (—½, 1¼, —2½) B (—2¾, 1, —¾).

2. Draw the top and front views of a line which passes through the point P(—2, ½, —2¼) and makes 45° with the line A(—1, 1¼, —⅜) B(—1, ¼, —1¾).

3. Find the distance between the two parallel lines A(—¾, 1⅜, —2¼) B (¾, ⅝, —¾) and C (½, 1¾, —2) E (2, 1, —½).

4. Find a point which is 1½" from A (—2½, ½, —2) and lies on the line B (—2, 1¼, —¾) C (—½, 0, —1¾).

5. Draw the top and front views of a square with center at C (¾, 1¼, —1¼) and side along the line A (—½, ¾, —1¼) B (1½, 1, —2).

6. Draw the top and front views of an equilateral triangle with center at C (¾, 1¼, —1¼) and side along the line A(—½, ¾, —1¼) B (1½, 1, —2).

7. Draw the top, front, and left side views of the square with center at P (1, 1¼, —1¼) and side along the line A (1¾, ⅞, —2) B (1¾, 2, —½).

8. Draw the top and front views of a 1¼" square with one side along the line A (—2, 1¼, —1⅝) B (—½, 2, —2½) and an adjacent side passing through the point P (—½, ⅝, —1⅜).

Use 7½" x 11" rectangles for each of the following problems.

9. Draw three views of a square with center at P (2, 2, —2) and side along the line A (3¼, 1, —3¼) B (3¼, 3½, —1½).

10. A 2" x 2½" rectangle has its center on the line O (¼, ½, —4½) P (2, 3¼, —2) and lies in a plane which contains the line

M (—½, ½, —2¾) N (3, 2¼, —2¾) and slopes downward to the back making 45° with H. Draw the top and front views of the rectangle when one side makes 30° with MN.

11. F (—½, 2½, —2) F (—3, 2½, —¾) is one line of a plane which makes 45° with V, sloping upward and backward. Draw the top and front views of an equilateral triangle (length of side 2¼″) lying on this plane, one corner at A (—1½, 2, ?), and one side making 30° with a horizontal line of this plane.

12. Draw the top and front views of a square lying on the plane A (0, 1½, —1½) B (—3, 3½, —1½) C (—3, 1½, —3½). One corner of the square is at O (—2, ?, —2) and an adjacent corner ¼″ higher and 1″ farther back than the point A.

13. S (0, ½, —2) S (3, 2½, —2) is a line of a plane which slopes downward to the front making 45° with H. Draw the top and front views of a 1¾″ square on this plane with center on the line C (1½, ½, —2¼) E (1½, 3, —3) and one corner ½″ further back than the center.

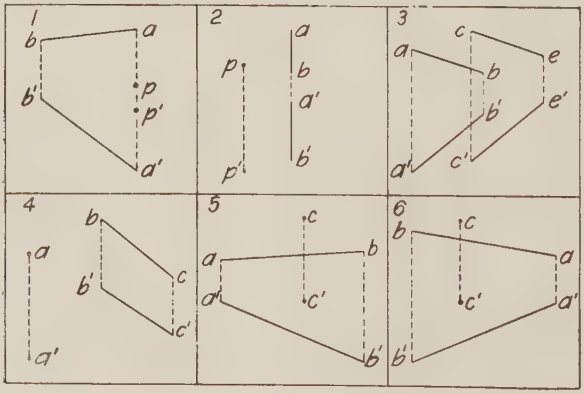
14. Draw the top and front views of an equilateral triangle which has its base along the line A (—3, 2, —3½) B (—½, 0, —1) and its vertex in the line C (—2, 3¼, —4¼) E (0, 1½, —2¾) and ½″ farther back than E.

15. S (0, ½, —2) S (3, 2½, —2) is a line of a plane which slopes downward to the front making 45° with H. Draw the top and front views of a 1¾″ square on this plane with center on the line C (1½, ½, —2¼) E (1½, 3, —3) and one side making 30° with the line S S.

16. S (½, 1, —3) S (3½, 2½, —3) is a line of a plane which slopes downward to the front making 45° with H. Draw the top and front views of an equilateral triangle lying on this plane with center at C (1½, 2¼, ?) and one side ¾″ farther back than C.

17. S (0, ½, —2) S (3, 2½, —2) is a line of a plane which slopes downward to the front making 45° with H. Draw the top and front views of a square lying on this plane with center at C (1½, 1¼, ?) and one corner 1″ farther back than C, length of side 1¾″.

18. A drain runs N 45° E on a falling 20% grade from the point P (—1¾, 0, —3). Draw the top and front views of the shortest connection to this drain from the point O (—¾, 2, —2½). What is the grade of the connection and how much pipe is required? Scale 1″=20′.



PERPENDICULAR RELATIONS BETWEEN LINES AND PLANES

81. Geometric proof that the top view of a line which is perpendicular to a plane is perpendicular to the top view of a horizontal line on that plane.

Let MN, Fig. 52, be the given line which is perpendicular to the given plane T. *mn* is the top view of MN or its orthographic projection on the plane H. SS is a horizontal line on plane T and coincides with its own top view.

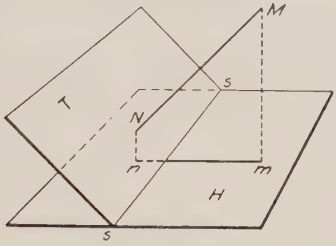


FIG. 52.—*Line MN perpendicular to plane T. Line mn perpendicular to line ss.*

Proof. The plane NMm is perpendicular to plane T, since MN is perpendicular to T.

The plane NMm is perpendicular to H, since Mm is perpendicular to H.

Since plane NMm is perpendicular to both T and H, it is perpendicular to their line of intersection SS.

Since SS is perpendicular to the plane NMm, it is perpendicular to the line *mn* which lies in this plane.

Therefore *mn* is perpendicular to *ss*. Q. E. D.

In a similar manner it can be proven geometrically that the front view of a perpendicular to a plane is perpendicular to the front view of a frontal line on that plane.

82. To draw the top and front views of a line which passes through a given point and is perpendicular to a given plane.

Let P, Fig. 53, be the given point and ABC the given plane.

Analysis. Since the line is to be perpendicular to the plane it will be perpendicular to every line of the plane. Although the line is perpendicular to every line of the plane it will appear perpendicular to horizontal lines only in the top view and to frontal lines only in the front view. The principle is general that two lines which are perpendicular to each other will appear perpendicular in any view in which one of the lines is shown in its true length.

Construction. Draw the horizontal line SS, its front view first and then its top view. From p draw po perpendicular to $s\ s$. Draw the frontal line FF, its top view first and then its front view. From p' draw $p'\ o'$ perpendicular to $f'\ f'$. Then $p\ o$ is the top and $p'\ o'$ the front view of the perpendicular to plane ABC from point P.

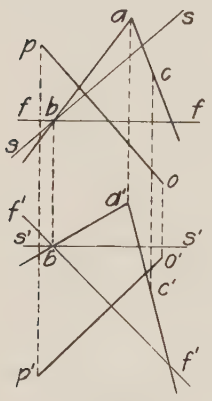


FIG. 53.—*Line OP perpendicular to plane ABC.*

In all the problems dealing with lines and planes at right angles to each other, the planes may be represented by horizontal and frontal lines unless the problem states definitely that they are to be represented otherwise. If the plane were not represented by its horizontal and frontal lines, these lines could easily be drawn, but the additional lines would complicate the drawing. Another advantage of representing a plane by its horizontal and frontal lines is that its position in space can more easily be visualized.

83. Given the top and front views of a point and of two lines which represent a plane, find the distance from the point to the plane.

Let SS and FF, Fig. 54, represent the plane and let P be the given point.

Analysis. Draw a perpendicular from the given point to the given plane, and find where it pierces the plane. The true length of the perpendicular from the given point to

the piercing point is the required distance. An auxiliary view which shows the plane as a line will locate the point in which the perpendicular pierces the plane and will show the true distance from the point to the plane.

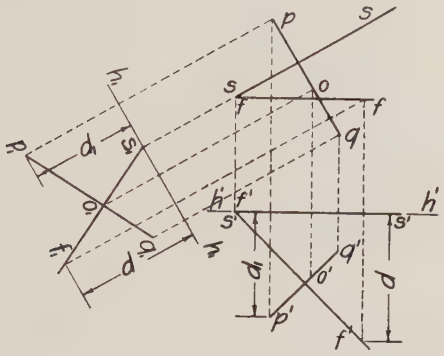


FIG. 54.—$p_1\ o_1$ *distance from point P to plane SSFF.*

Construction. In the auxiliary view taken in the direction of the line SS, $s_1\ f_1$ represents the given plane and p_1 the given point. $p_1\ o_1$, perpendicular to $s_1\ f_1$, is the true distance from the point to the plane. $p\ q$, perpendicular to ss, and $p'q'$ perpendicular to $f'f'$, are the top and front views respectively of the perpendicular to the plane from the point P; O is the point in which this perpendicular pierces the plane.

84. Conversely, to find the top and front views of a point which is a given distance from a given plane.

Let SS and FF, Fig. 55, represent the given plane.

Analysis. All the points which are a given distance, say 1″, from a given plane SSFF lie in a plane parallel to and 1″ from the given plane. There are two such planes, one on each side of the given plane. An auxiliary view can be taken which shows these planes as three parallel lines, 1″ apart. From the auxiliary view, the top and front views of a point 1″ from the plane SSFF can be found.

Construction. In an auxiliary view taken in the direction of SS, the plane is represented by s_1 f_1. a_1 b_1 and m_1 n_1 represent planes parallel to and 1″ from the given plane. Any point, as p_1, in either a_1 b_1 or m_1 n_1 can be

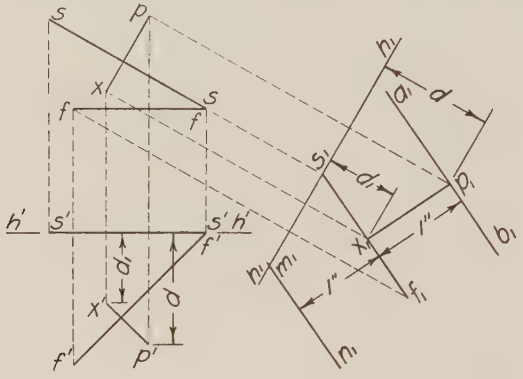


Fig. 55.—*Point P is 1″ from plane SSFF.*

taken as the auxiliary view of the required point. The top view p can be any point on the line p_1 p, parallel to s s. The distance d taken from the auxiliary view is used to locate p'. Then p and p' are the top and front views respectively of one of any number of points which are 1″ from the given plane SSFF.

X is the foot of a perpendicular from the point P to the plane. If a 1″ cube with its base in the plane SSFF had one corner at X, then the point P would have been a corner in the other base of the cube. x p is perpendicular to s s and x' p' perpendicular to f' f'.

85. Given the top and front views of a point and of an oblique line. Represent a plane which contains the point and is perpendicular to the line.

Let P, Fig. 56, be the given point and AB the given line.

Analysis. Since the plane is to be perpendicular to the line, a horizontal line of the plane will have its top view

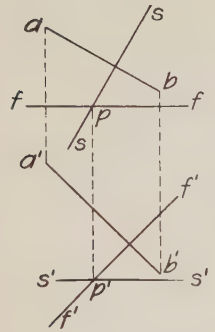


FIG. 56. — *Plane SSFF perpendicular to line AB.*

perpendicular to the top view of the line and its front view parallel to G. L. Likewise, a frontal line of the plane will have its front view perpendicular to the front view of the line and its top view parallel to G. L. Therefore to represent the plane, draw the top and front views of a horizontal line and of a frontal line which pass through the given point and are perpendicular to the given line. These two lines will represent the required plane.

Construction. Through p, draw ss perpendicular to ab and through p' draw s' s' parallel to G. L. Through p', draw f' f' perpendicular to a' b' and through p draw ff parallel to G. L. Then the plane of SS and FF is perpendicular to the line AB.

Although the plane of SS and FF is perpendicular to the line AB, the line AB does not intersect either of the lines SS or FF.

86. Problems.

1. Find the distance from the point P (¼, 1⅞, —¾) to the plane A (—2¼, ½, —¾) B (—¼, 1½, —2¼) C (1¼, 0, —1¾).

2. Find the distance from the point O (—½, 1¾, —½) to the plane of A (—2, 1½, —2) B (½, 1½, —2) and C (—2, ¼, —1) E (½, ¼, —1).

3. O (—1⅝, ⅜, —2) is one corner of a cube which has its base in the plane C (—2¾, 1½, —½) A (—½, 1¾, —2¼) B (—½, ½, —¾). Find the size of the cube.

4. B is one corner of a 1½" cube which has its base in the plane A (—2½, 1, —2¼) B (—1, 1¾, —¾) C (½, ½, —1¾). Locate an adjacent corner which does not lie in the plane of the base.

5. Represent a plane which is parallel to and is 1" from the plane A (—1½, 1, —½) B (—⅜, ½, —1¾) C (1¼, 2, —¼).

6. Locate a point which is 1" from the plane A (—¾, 1⅜, —2¼) B (¾, ⅝, —¾) C (½, 1¾, —2) E (2, 1, —½.)

7. Represent a plane which contains the line M (¼, 1, —¾) N (1¾, 2, —2¼) and is perpendicular to the plane A (—1¾, 2, —3) B (—¼, 3½, —2) C (1¼, 2½, —4). (Use a 7½" x 11" rectangle for this problem).

8. Find the locus of all points which are equidistant from the points A (—1½, 1, —2) B (0, 3, —3) and C (1½, 2, 0). (Use a 7½″ x 11″ rectangle for this problem.)

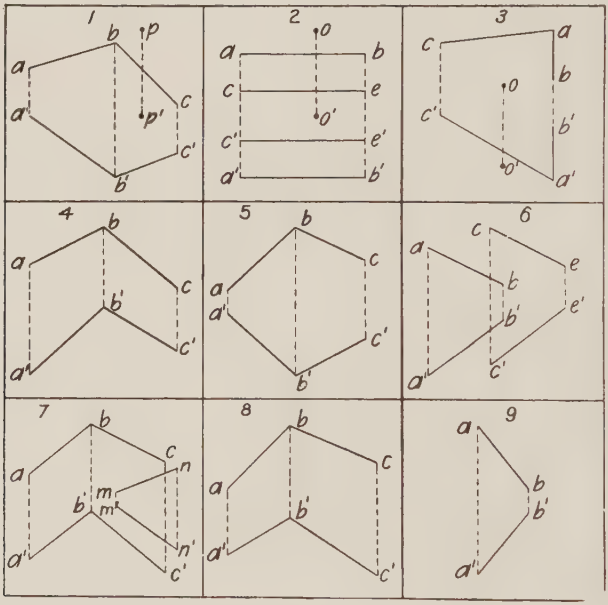
9. A (—1, 1¾, —2¼) B (0, ¼, —¾) is one edge of a cube. Represent by two lines the plane of the base of the cube.

10. Find the center of the circle in which P (—½, ½, —1) moves when it rotates about A (0, 1¾, —2¼) B (1, ¼, —¾) as an axis.

11. A (—2, 2¼, —3) B (—1, ½, —1¼) is the base of an isosceles triangle which has its vertex in the line C (½, ½, —3) E (2, 3¾, —2½). Represent the triangle in the top and front views. (Use a 7½″ x 11″ rectangle for this problem.)

12. Find the top and front views of a point O which is equidistant from A (—¾, ½, —½) B (¾, ½, —1½) C (¾, 1½, —½) and E (—¼, 1½, —1).

13. Draw the top and front views of the line MN which intersects the lines A (½, 1, —1½) B (1½, ½, —1¾) and A (½, 1, —1½) C (0, ½, —2) and is parallel to and 1″ from the plane E (0, 0, —1) F (1½, 0, —2) G (1½, 1, —1).



87. Given the top and front views of two lines of a plane and the top and front views of a third line which does not lie on this plane. Find the projection of the third line on the given plane.

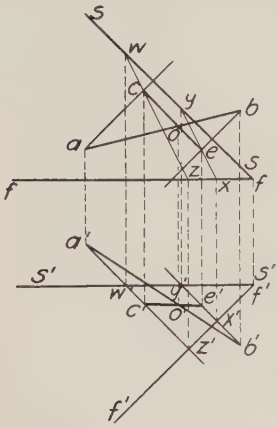


FIG. 57.—*CE projection of AB on plane SSFF.*

Let SSFF, Fig. 57, represent the given plane and AB any line which does not lie on this plane.

Analysis. From any two points of the given line, erect perpendiculars to the given plane. A line joining the points in which these perpendiculars pierce the plane will be the required projection. The point in which the given line pierces the given plane is also a point on the projection of the line on the plane.

Construction. Draw *a c* and *b e* perpendicular to *s s* and *a' c'* and *b' e'* perpendicular to *f' f'*. These are the top and front views respectively of perpendiculars to the plane SSFF from A and B and pierce the plane at C and E. *c e* is the top and *c' e'* the front view of the projection of the line AB on the plane SSFF. AB and CE intersect at O, the point in which AB pierces the plane SSFF.

88. Problems.

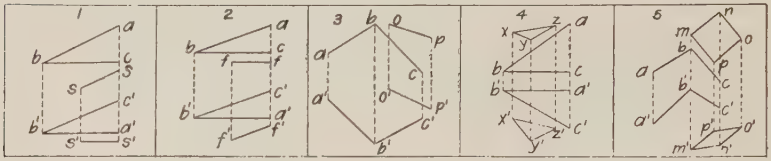
1. Find the projection of the line S (½, 0, —2¼) S (1½, ½, —2¼) on the plane A (1½, 2⅛, —2) B (—1¼, ¾, —2) C (1½, ¾, —⅜).

2. Find the projection of the line F (½, ¼, —2¼) F (1½, ¼, —1¾) on the plane A (1½, 2⅛, —1½) B (—1¼, ½, —1½) C (1½, ½, —⅛).

3. Find the projection of the line O (⅛, 1¾, —½) P (1, 1⅜, —1⅜) on the plane A (—1, ¾, —¾) B (—¼, 1½, —1¾) C (¾, ¼, —1¼).

4. Find the projection of the triangle X (—¾, 1¾, —1½) Y (0, 1½, —2) Z (½, 2, —1¾) on the plane A (1, 2, —½) B (—1, ¼, —½) C (1, ¼, —2).

5. Find the projection of the parallelogram M (0, 1¾, —2½) N (¾, 2, —1½) O (1½, 1½, —1½) P (¾, 1¼, —2½) on the plane A (—1¼, ½, —2) B (0, 1½, —½) C (1¼, 0, —1).



89. Given the top and front views of two lines of a plane and the top and front views of a third line which does not lie on this plane, find the angle which the third line makes with the given plane.

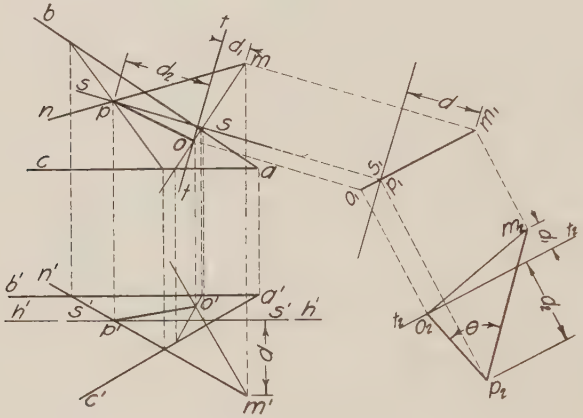


FIG. 58.—θ *is angle line MN makes with plane ABC.*

Let ABC, Fig. 58, represent the given plane and MN any line which does not lie on this plane.

Analysis. The angle which a line makes with a given plane is understood to be the angle which the line makes with its projection on that plane. Therefore to find the angle which a line makes with a given plane, find the projection of the line on the plane and then find the angle between the line and this projection.

Construction. The line MN pierces the plane ABC at P. A perpendicular to the plane from M pierces it at O. Then OP is the projection of MN on the plane ABC. To find the true size of the angle MPO, picture these points from their top views and then take an auxiliary view in the direction of the line SS. m_1 p_1 o_1 represents the triangle in this view. After picturing the triangle MPO over the first auxiliary view, take a second auxiliary view with the line of sight perpendicular to the plane of the figure, using the distances d_1 and d_2 from the top view to locate m_2 and p_2 in the second auxiliary view. m_2 p_2 is the true length of the line MP, p_2 o_2 the true length of the projection of MP on the lane ABC, and m_2 o_2 the length of the perpendicular from M to the plane ABC. Therefore m_2 p_2 o_2 is the true size of the angle which the line MN makes with the plane ABC.

90. Problems.

1. Find the angle which the line A (1¾, 0, —2½) B (2¼, 1¾, —½) makes with the plane S (1, ¼, —2) S (3¼, 1¾, —2) F (3¼, ¼, —¾).

2. Find the angle which the line O (1⅝, 1¾, —½) P (2½, 1⅜, —1⅜) makes with the plane A (½, ¾, —¾) B (1¼, 1½, —1¾) C (2¼, ¼, —1¼).

3. Find the angle which the line A (—2, 1⅛, —2⅛) B (0, 1⅛, —2⅛) makes with the plane M (—1½, ¼, —1⅛) N (—1, 1¾, —⅝) O (—1, ½, —2¼).

91. Two oblique lines which do not lie in the same plane are given by their top and front views, find the top and front views and the true length of their common perpendicular.

General analysis. A line can be drawn which is perpendicular to each of two given lines and cuts them both. Since the required line is to be perpendicular to each of the given lines it will show at right angles to one of them in any view in which one of the given lines or the perpendicular itself shows true length. Consequently to draw the perpendicular, take an auxiliary view which shows one of the given lines as a point. In this view, the perpendicular

will show true length and will therefore appear at right angles to the line which does not appear as a point. From this view other views of the perpendicular can be found. If one of the given lines is parallel to either H or V, but one auxiliary view is necessary to show this line as a point, Case 1. If both of the given lines are oblique to H and V, two successive auxiliary views are necessary to show one of the lines as a point, Case 2.

92. Case 1. When one of the lines is parallel to either H or V.

Let AB and SS, Fig. 59, be the given lines.

Analysis. Take an auxiliary view of the lines by looking in the direction of the line which is parallel to H. In the auxiliary view, this line appears as a point and the

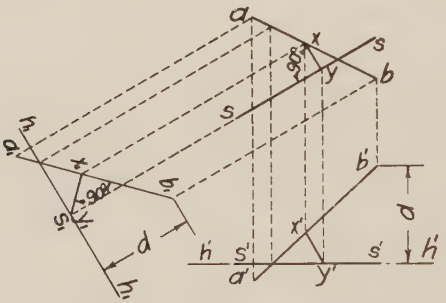


FIG. 59.—*XY common perpendicular to AB and SS.*

other one as a line. A perpendicular from the point to this line is the auxiliary view of the common perpendicular to the two given lines, and is the true length of the perpendicular. From the auxiliary view of the perpendicular, its top and front views can be found.

Construction. s_1 and a_1 b_1 are the auxiliary views of the given lines taken by looking in the direction of SS; h_1 h_1 represents the horizontal plane at the level of SS. x_1 y_1, perpendicular to a_1 b_1 is the auxiliary view of the common perpendicular and is its true length. x is the top view of the point X. xy, perpendicular to ss, is the top

and x' y' the front view of the required perpendicular to the lines AB and SS.

When one of the given lines is parallel to V instead of H as above, the auxiliary view should be taken by looking in the direction of this line.

93. Case 2. When both of the lines are oblique to H and V.

Let AB and CE, Fig. 60, be the given lines.

Analysis. Take a view of both lines by looking parallel to H and perpendicular to one of the lines. Take a second auxiliary view by looking in the direction of the line which

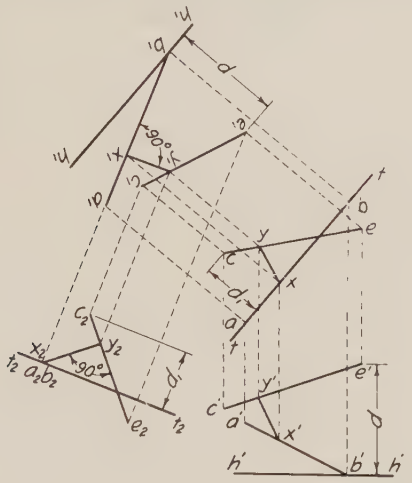


FIG. 60.—*XY common perpendicular to AB and CE.*

is shown in its true length in the first auxiliary view. In the second auxiliary view, one line appears as a point and the other as a line. From the point draw a perpendicular to the line. This is the second auxiliary view of the required perpendicular and shows its true length. From this view the first auxiliary, the top, and the front views of the perpendicular can be drawn.

Construction. Picture the lines AB and CE over their top views. The first auxiliary view is taken by looking

parallel to H and perpendicular to the line AB. a_1 b_1 and c_1 e_1 are the auxiliary views of the given lines. The distance d is taken from the front view and used to locate e_1 in the auxiliary view. Picture the lines AB and CE over the first auxiliary view. Take the second auxiliary view by looking in the direction of AB. In this view AB appears as the point a_2 b_2 and CE appears as the line c_2 e_2. The distance d_1 is taken from the top view and used to locate c_2 in the second auxiliary view. From the point a_2 b_2 draw the line x_2 y_2 perpendicular to the line c_2 e_2. x_2 y_2 is the second auxiliary view of the common perpendicular to the two lines and is its true length. From the point y_2, draw a line parallel to a_1 b_1 intersecting c_1 e_1 at y_1. Draw y_1x_1 perpendicular to a_1 b_1. x_1 y_1 is the first auxiliary view of the common perpendicular. From this view the top view xy and then the front view x' y' of the common perpendicular can be drawn.

94. Problems.

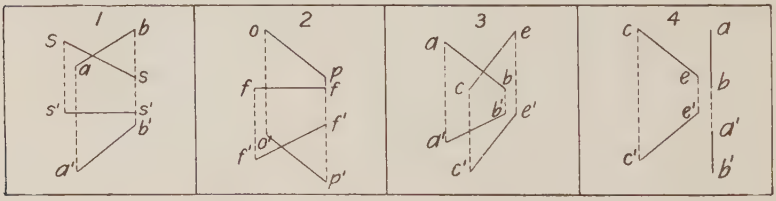
1. Draw the top and front views of the common perpendicular to the lines S (—1½, 1¼, —1) S (0, ½, —1) and A (—1¼, ¾, —2¼) B (0, 1½, —1¼).

2. Draw the top and front views of the common perpendicular to the lines F (—1½, ½, —2) F (0, ½, —1¼) and O (—1¼, 1¾, —1½) P (0, ¾, —2¼).

3. Find the top and front views of the common perpendicular to the lines A (—¼, 1¾, —1⅝) B (1, ¾, —1) and C (¼, ¾, —2¼) E (1¼, 2, —1).

4. Find the top and front views of the common perpendicular to the lines A (¼, 1½, —1½) B (¼, ¼, —2¼) and C (—¾, 1½, —2) E (0, ½, —1¼).

5. Find the length and location of the shortest connection which can be made between two pipes with center lines A (—1⅛, ¾, —2) B (—⅜, 0, —2¼) and C (—⅞, ½, —1¼) E (0, ⅝, —2¼). (Use center lines only).



GENERAL PROBLEMS IN POINT, LINE, AND PLANE.

MISCELLANEOUS PROBLEMS

In the following list of problems, those marked (*a*) can be conveniently solved in a 5½" x 7½" rectangle, those marked (*b*) in a 7½" x 11" rectangle, and those marked (*c*) in a 11" x 16" rectangle. In each case the base line should be drawn through the center of the rectangle and in case (*a*) parallel to the longer side, and in cases (*b*) and (*c*) parallel to the shorter side.

1. (*a*) A (—½, 0, —½) B (½, ½, ?) C (?, ?, —1½) is a 1½" equilateral triangle. Draw its top and front views.

2. (*a*) The top views of two points are ¾" apart and their front views are 1½" apart. What is the greatest and what is the least distance apart that the points can be placed?

3. (*a*) Draw the top and front views of the locus of a point which is 2" from A (—1¼, ¼, —1) and 1½" from B (¼, ¾, —2).

4. (*a*) A ray of light passes through P (—2, 1½, —1¼) and is reflected from V at the point O (0, 0, —½). Find the top and front views of the point in which the reflected ray strikes H.

5. (*a*) Find the top and front views of a ray of light which emanates from A (—2, 1, —1½) and after being reflected from H and V passes through B (1, ½, —1¾).

6. (*a*) Draw the top and front views of the ray of light given in problem 5 after it has been reflected from H, V, and a profile plane through B. What is the direction of the ray after being reflected from the profile plane with respect to the direction of the ray as it emanates from A?

7. (*a*) A ray of light from A (—1½, 1¼, —1⅞) is reflected at B (0, 1⅞, —1¼) to C (—½, ¼, —2). Represent the reflecting plane in the top and front views.

8. (*a*) Find the top and front views of a point O, which joined with the points A (0, 2, —1¾), B (—1½, ½, —1¾), and C (1, 0, —1¾) makes the angles AOB, BOC, and COA all right angles.

9. (*a*) Represent a plane which contains the line A(—2, ½, —1) B (—½, 1½, —1⅝) and is in such a position that AB bisects the angle between a horizontal and frontal line of the plane. Draw the horizontal and frontal lines through A.

10. (*a*) Find the top and front views of the shortest line which is parallel to H and terminates in the lines A (—¾, 2, —1) B (¼, ½, —2) and C (½, 1½, —1½) E (—1½, ⅝, —⅞).

11. (*b*) Cut a parallelogram having one side 2" long from the pyramid O—ABCE. O (1½, 3½, 0) A (—2⅛, ½, —4) B (½, —½, —4) C (2½, 1½, —4) E (—1⅜, 3½, —4.)

Mining Problems

In the following list of problems, those marked (*a*) can be conveniently solved in a 5½" x 7½" rectangle, those marked (*b*) in a 7½" x 11" rectangle, and those marked (*c*) in a 11" x 16" rectangle. In each case the base line should be drawn through the center of the rectangle and in case (*a*) parallel to the longer side, and in cases (*b*) and (*c*) parallel to the shorter side.

1. (*b*) At a point of outcrop O (0, 1, —1) the direction of strike of a body of ore is N. 45° E. The "dip" (the inclination of the face of the ore body to the horizontal) is 60° S. E. A tunnel is driven at P (—1, 2, —3) S. 60° E. on a rising 10% grade. How far must the tunnel be driven to reach the ore body?

NOTE. The "direction of strike" is the direction of a horizontal line in the plane of the face of the ore body.

2. (*b*) Two ore veins have directions of strike due N. E. and both veins dip 30° S. E. The strike lines at the same elevation are 200′ apart. What is the shortest distance between the veins?

3. (*b*) From the point of outcrop A (0, 3, 0), run S. 30° W. 100′ to a drillhole B. The drillhole C is 100′ from both A and B. The drill strikes ore at B at a depth of 25′ and at C at a depth of 50′. Determine the direction of strike and dip of the vein. Scale 1"=25′.

4. (*b*) The direction of strike of a vein passes through the origin and bears N. 75°E. The vein dips 60° toward the north and west. A drillhole which passes through the origin, bears N. 30° E. and dips 75°. If the core from the drillhole shows 6′ of vein matter, what is the true thickness of the vein? Scale ½"=1′—0". (Omit the front view.)

5. (*b*) The direction of strike of a vein passes through the origin and bears N. 45° E. A drillhole through the origin bears N. 15°E. and dips 75°. If the core from the drillhole shows 6′—0" of vein matter and the vein is known to be 3′—0" thick, what must be the dip of the vein? Scale ½"=1′—0". (Omit the front view).

6. (*b*) A vein of ore strikes N. 30° E. from the point A (—1, 0, ?) and dips 60° S. E. A N. and S. tunnel 100′ lower than the point A intersects this vein. What is the apparent dip of the vein on the side wall of the tunnel? Omit the front view. Scale 1"=50′.

7. (*b*) Two holes are drilled close together on a N. and S. line. The first hole, drilled at the point A (—2½, 1¼, —1½), is vertical and intersects the bedding plane of a rock at an angle of 45°. The second drill hole which bears N. and dips 60°, intersects the same bedding plane at an angle of 30°. Determine the strike and dip of the bedding plane. (Base line parallel to longer sides of rectangle.)

8. (*b*) A rich ore pocket is found at the point O (0, 1½, ?) which is 75′ below the surface of the earth. This pocket was cut in two by a fault plane which strikes N. 30° E. and dips 60° S. E. The portion of the vein S. E. of the fault plane was found to have moved 100′ in a S. E. direction, measured on the horizontal plane and perpendicular to the strike line of the fault plane. The scratches on the fault plane were found to dip 45° in a southerly direction. Determine on the top view the location and depth of a vertical shaft to reach the other portion of the ore pocket. Scale 1″=100′. (Omit the front view).

9. (*b*) A rich ore pocket is found at the point O (0, 1½, ?) which is 100′ below the surface of the ground. This pocket is cut in two by a fault plane which strikes N. 30° E. and dips 60° S. E. This pocket lies on a vein N.W. of the fault plane; the direction of strike of the vein is N. 30° W. and the dip is 60° N. E. The portion of the vein on the 100′ level, S. E. of the fault plane, was found to have moved 100′ in a S. W. direction measured on a horizontal plane, and perpendicular to the strike line. The scratches on the fault plane are found to dip 45° in a southerly direction. Determine the location and depth of a vertical shaft to reach the lost portion of the ore pocket. Scale 1″=100′. (Omit the front view).

10. (*b*) A rich ore chute lies in a vein whose strike is E. and W. through the point O (0, 2, ?) and whose dip is 60° S. A shaft following this chute of ore starts at O and has a dip or plunge of 30° in a S. E. direction. 600′ from O the shaft meets a fault plane whose strike is N. 30° E. and whose dip is 45° N. W. The scratches on the fault plane pitch S. W. and make 60° with the direction of strike. If the vein is offset 150′ to the south by faulting, find the location of the remainder of the ore chute. (Scale 1″=300′. G. L. parallel to longer side of 7½ x 11 sheet). (Omit the front view).

Cube Problems

In the following list of problems, those marked (*a*) can be conveniently solved in a 5½″ x 7½″ rectangle, those marked (*b*) in a 7½″ x 11″ rectangle, and those marked (*c*) in a 11″ x 16″ rectangle. In each case the base line should be drawn through the center of the rectangle and in case (*a*) parallel to the longer side, and in cases (*b*) and (*c*) parallel to the shorter side.

1. (*b*) Given a plane A (—2, ¼, 0) B (½, ¼, —2½) C (½, 2¾, 0) and a point O (—1⅝, 1½, —1¼). Draw the top and front views of a cube which has one corner at O, its base in the plane ABC, and one corner of the base in the line AC.

2. (*b*) Draw the top and front views of a 1¼″ cube with one edge along the line A (¾, 1⅞, —⅞) B (¾, ⅛, —2⅝) and an adjacent edge passing through the point P (0, 1⅝, —1⅝).

3. (*b*) Draw the top and front views of a 1¼″ cube standing on the plane A (0, 1¾, —1) B (0, 0, —2¾) P (—¾, 1½, —1¾). One side of the base is to make 30° with AB.

4. (*b*) A plane which makes 30° with H and slopes downward to the right has S (—2¼, —⅛, —2⅜) S (0, 2⅛, —2⅜) for a horizontal line. C (½, 1, —1⅝) is an upper corner of a cube which has its base in the given plane. One corner of the base is to be on the horizontal line SS. Draw the top and front views of the cube.

5. (*b*) Given a plane A (—2, —⅛, —⅜) B (½, —⅛, —2⅞) C (½, 2⅜, —⅜). Draw the top and front views of a 1¼" cube which has its base in the given plane. One corner of the base is to be on the line AC at the point O (—½, ?, ?) and one edge of the base is to make 30° with the line AC.

6. (*b*) S (—⅞, 2½, —1¾) S (¼, 1½, —1¾) is one line of a plane which makes 30° with H. Draw the top and front views of a 1¼" cube which has its base in the plane. One side of the base is to make 30° with SS.

7. (*b*) Given a plane A (—2, —⅛, —⅜) B (½, —⅛, —2⅞) C (½, 2⅜, —⅜). Draw the top and front views of a 1¼" cube which has its base in the given plane. One corner of the base is at M (—½, ?, —⅝) and another corner of the base is ½" lower than the line AC.

8. (*b*) Given a point P (—⅞, 1⅞, —2) and a line A (—1, 1, —1½) B (—⅛, 1⅜, —2). Draw the top and front views of a cube having AB for one edge and the plane ABP for the plane of the base.

9. (*b*) Given a plane M (—1, 1, 0) N (½, 2½, 0) O (½, 1, —1½). A 1¼" cube rests on this plane in such a position that two edges AB and EF of the base make angles of 15° with the horizontal line MN. The corner B is at the point (—½, ?, —½). Draw the top and front views of the cube.

10. (*b*) S (—1⅜, 2½, —1¾) S (¼, 1½, —1¾) is one line of a plane which makes 30° with H. Draw the top and front views of a 1¼" cube which has its base in the plane. One side of the base is to make 30° with a frontal line of the plane.

11. (*b*) One edge of the base of a 1¼" cube lies along the line A (—1⅜. ⅞, —1⅜) B (¼, 1½, —2¼) and an adjacent edge passes through the point P (—⅜, 2, —2¼). Draw the top and front views of the cube.

12. (*b*) A (—½, 1, ?) B (⅛, 1½, ?) is a line which makes 45° with H. This line is one edge of a cube. Draw the top and front views of the cube when none of the edges are parallel to either H or V.

13. (*b*) A (—½, 1, ?) B (⅛, 1½, ?) is a line which is 1¼" long. This line is one edge of a cube. Draw the top and front views of the cube when none of the edges are parallel to either H or V.

14. (*b*) M (—1½, 1, ?) N (—1, 1½, ?) is a line which makes 30°, 45°, or 60° with H. Draw the top and front views of a cube having MN for one edge. One corner of the base is to be ¼" lower than M.

15. (*b*) Draw the top and front views of a cube with one corner at A (0, 1½, —1), another at B (1, ⅝, —1⅜), and a side along the line joining A with C (—1, 1, ?).

16. (*b*) C (—⅝, 1⅛, —1) E (¾, 2, —2⅜) is a diagonal of a cube. One corner of the cube is in a plane which makes 60° with H and contains a horizontal line passing through P (—1⅞, ¼, —¼) and making 45° with V. Draw the top and front views of the cube.

17. (*b*) C (—1⅝, ⅝, —⅝) E (⅝, ⅝, —⅝) is a diagonal of a cube which has one corner in a horizontal plane through the origin. Draw the top and front views of the cube.

18. (*b*) One edge of a 1¼" cube lies along the line A (—1⅛, 1⅝, —2) B (⅝, 0, —1⅜) and an adjacent edge passes through the point P (¼, ¾, ?) in such a direction that its top view makes 60° with the top view of AB. Draw the top and front views of the cube.

19. (*b*) Draw the top and front views of a 1½" cube with one corner at O (0, 1½, —1) and three edges along the lines joining O with the points A (—1, 1, ?) B (⅞, 3, ?) and C (1¼, ½, ?).

Pyramid Problems

In the following list of problems, those marked (*a*) can be conveniently solved in a 5½" x 7½" rectangle, those marked (*b*) in a 7½" x 11" rectangle, and those marked (*c*) in a 11" x 16" rectangle. In each case the base line should be drawn through the center of the rectangle and in case (*a*) parallel to the longer side, and in cases (*b*) and (*c*) parallel to the shorter side.

1. (*b*) O (—1, 2⅛, —½) is the vertex of a right square pyramid, which has its base on the plane A (—2, ¼, 0) B (½, ¼, —2½) C (½, 2¾, 0). One side of the base is to make 30° with the line AB. Draw the top and front views of the pyramid with side of base 1⅛".

2. (*b*) One edge of the base of a right square pyramid lies along the line A (¾, 1¾, —1) B (¾, 0, —2¾) and an adjacent edge passes through the point P (0, 1½, —1¾). Draw the top and front views of the pyramid with side of base 1¼" and altitude 1¾".

3. (*b*) One side of an equilateral triangle lies along the line A (—¼, 1⅞, —1¼) B (1⅞, ¾, —2⅞) and the vertex opposite this side is at the point C (1½, 2½, —1½). Draw the top and front views of a regular pyramid having this triangle for a base and an altitude of 2½".

4. (*b*) P (—¼, 1¾, —1¼) is the vertex of a right square pyramid which has its base in the plane A (—1½, ½, —1¼) B (½, ½, —2⅜) C (—1½, 1⅝, —2⅜). One side of the base is to make 30° with AB. Draw the top and front views of the pyramid when the side of the base is equal to the altitude.

5. (*b*) Given a plane A (—2, ¼, 0) B (½, ¼, —2½) C (½, 2¾, 0) and a point O (—1, 2⅛, —1½). Draw the top and front views of a right square pyramid with vertex at O and base in the plane ABC. The side of the base is to be 1⅜", and one corner of the base to be in the line AC.

6. (*b*) The base of a right pyramid whose altitude is 1¾" is an equilateral triangle with one side along the line A (—2¼, ⅛, —2⅛) B (⅛, ¾, —½) and one corner on the line M (—2¼, 1¼, —3¼) N (⅛, 1¾, —1½). Draw the top and front views of the pyramid.

7. (*b*) Draw the top and front views of a right square pyramid with vertex at O (—1¼, 1⅜, —2¼) and base in the plane A (—2¼, —½, —¾) B (¼, —½, —3¼) C (¼, 2, —¾). One corner of the base is to be on the line AC, side of base 1⅜″.

8. (*b*) P (—½, ⅞, —1⅞) is the center of an equilateral triangle having one side along the line A (—2¼, 0, —2¾) B (⅝, ¾, —¾). This triangle is the base of a regular pyramid with an altitude of 1¾″. Draw the top and front views of the pyramid.

9. (*b*) Draw the top and front views of a right square pyramid which has its base on the plane A (—2, —½, —1) B (0, —½, —3) C (0, 1½, —1). One corner of the base is on the line AC, at the point O (—½, ?, ?) and one side of the base makes 30° with the line AC. Side of base of pyramid 1¼″, altitude 1½″.

10. (*b*) a (—1½, 1⅛, ?) b (—¾, 1¾, ?) is the top view of a line which is 1¾″ long. Draw the top and front views of the right square pyramid having AB for an axis. One side of the base is to make 30° with a frontal line on the plane of the base. Side of base 1¼″.

11. (*b*) A (—1½, 1⅛, ?) B (—¾, 1¾, ?) is a line which makes 60° with H. Draw the top and front views of the right square pyramid having AB for an axis. One corner of the base is to be ⅜″ farther back than B, side of base 1¼″.

12. (*b*) A regular triangular pyramid whose altitude is 2½″ stands on a plane which contains M (—1, 2, —2⅛) N (1⅜, ⅝, —2⅛) and slopes downward to the back making 45° with H. Draw the top and front views of the pyramid when one corner of the base lies in the line A (—1⅛, 1½, —1⅛) B (—⅜, ¾, —2¾) and one side of the base lies along the line of intersection of the plane of the base and a plane perpendicular to AB at a point ½″ from B.

13. (*c*) Draw the top and front views of a regular triangular pyramid standing on a plane through A (—1¾, 2¼, —3½) parallel to the lines B (—3½, —¾, —2¼) C (2¼, 3¼, —4) and E (—1¼, 3½, —2½) F (1¾, 0, —5¾). One side of the base lies in the line of intersection of the plane of the base and a plane through A perpendicular to the line BC. The vertex is at the point O (2¾, —¾, —¾). Find the points in which BC and EF pierce the pyramid.

The above problem can be varied by making the base a square, regular pentagon, regular hexagon, or changing the pyramid to a cone with the base tangent to the line of intersection of the two planes.

14. (*b*) Draw the top and front views of a triangular pyramid O-ABC. The base is A (—⅜, 1⅝, —2½) B (⅝, 1, —1⅝) C (1⅝, 2¼, —2⅛). The altitude is 1⅜″, while OA=1¾″ and OB=1½″ are two of the lateral edges.

15. (*b*) Draw the top and front views of a tetrahedron two edges of which are 2″, two 2½″, and two 3¼″. The edges are to be oblique to H and V.

16. (*b*) Draw the top and front views of three spheres, each tangent to the other two and having the following radii: 1″, 1½″, 2″. Draw the views of a fourth sphere with radius ⅝″ which will be tangent to the other three spheres. The plane of the centers of any three spheres is to be oblique to both H and V.

Prism Problems

In the following list of problems, those marked (*a*) can be conveniently solved in a 5½" x 7½" rectangle, those marked (*b*) in a 7½" x 11" rectangle, and those marked (*c*) in a 11" x 16" rectangle. In each case the base line should be drawn through the center of the rectangle and in case (*a*) parallel to the longer side, and in cases (*b*) and (*c*) parallel to the shorter side.

1. (*b*) A (—¼, 1, —1½) is one corner of the base of a right square prism which has its axis along the line B (0, ⅛, —1) C (2, 2, ?). The axis makes 30° with H, and is 2½" long. Draw the top and front views of the prism.

2. (*b*) In problem 1 make the base of the prism an equilateral triangle with one corner at A.

3. (*b*) The line B (—¾, ⅛, —1) C (1¼, 2, ?) makes 30° with H. Draw the top and front views of a right square prism with a 2½" axis lying along the line BC and the center of one base ¼" lower than B. The diagonals of the base are 1½" long and one of them is to make 30° or 45° with H as the instructor assigns.

4. (*b*) A (—2½, 1½, —1¾) B (—1¼, 2½, ?) is a line which is 2½" long. Draw the top and front views of a right square prism which has AB for an axis. One side of the base of the prism makes 30° with a frontal line on the plane of the base. Side of base 1¼".

5. (*b*) A (—2½, 1½, —1½) B (—1¼, 2½, ?) is a line which makes 60° with H. Draw the top and front views of a right square prism which has AB for an axis. One corner of the base is to be ⅞" farther back than B, diagonal of base 2".

6. (*c*) The common perpendicular to the lines A (—2, 4, —1½) B (—¾, ½, —4¼) and C (—3¾, 2½, —2½) E (—¾, 3¼, —4½) is the axis of a right rectangular prism. The diagonals of the upper and lower bases coincide with AB and CE respectively. The shorter edge of the base is 1¾" long. Draw the top and front views of the prism.

7. (*c*) Substitute A (2, ½, —2¼) B (2, 2½, —4¼) and C (—3½, 2¾, —2) E (¼, 5½, —½) in problem 6.

8. (*b*) Given the points O (¼, ⅜, —⅞), A (1⅜, 1, —1½), B (—⅝, ⅞, —1½), C (—1¼, 1⅞, —1½), and E (—1, 2⅞, —2½). Draw the top and front views of a parallelopiped with edges along the lines OA, OB, and OC, and a 2½" diagonal along OE.

CHAPTER IV

SHADES AND SHADOWS

95. In order that the views of an object shall more nearly represent the object as it appears, the shadows which the different parts of the object cast on other parts or on the planes of projection are sometimes shown. Shadows are used more in architecture than in any other branch of engineering but every engineer should know how to find the shadow of a simple object. Only the elementary principles of the subject will be outlined here.

96. Definitions. If an opaque object is placed between a surface and the source of light, the object will cast a shadow upon the surface. This is due to the fact that the object intercepts the rays of light and prevents their reaching the surface upon which the shadow is cast. When the light strikes an opaque body, that portion of the surface of the body which is turned away from the source of light and which, therefore, does not receive any of the light rays is said to be in the shade.

The source of light is the sun and because of its distance from the earth the rays of light are assumed to be parallel. The direction of the rays of light change with the time of day and consequently change the shadow of an object. For the sake of uniformity, the direction of a ray of light is taken as the direction of the diagonal of a cube which joins the upper, left hand, front corner with the lower, right hand, back corner when the cube has two faces parallel to V and two parallel to H. In other words a ray of light is considered as coming over the left shoulder of the observer so that both its top and front views make angles 45° with the ground line.

97. Shadows of points and lines. To find the shadow of a **point** on a surface, pass a ray of light through the

point and find where this ray pierces the surface. This piercing point is the required shadow point.

To find the shadow of a **straight line** on a plane, find the shadows of two points of the line on the plane and join them by a straight line. If the shadow of the straight line falls upon two or more planes, find the shadows of two points of the line on each plane and join the corresponding shadow points. The shadow of the straight line is a broken line which changes direction on the line of intersection of the planes upon which the shadow is cast.

To find the shadow of a straight line on a curved surface, find the shadows of several points of the line and join these shadow points in the proper order.

To find the shadow of a **curved line** on a surface, it is usually necessary to find the shadows of several points of the curve on the surface and then join the shadow points in the proper order.

The shadow of a **plane figure** on a plane which is parallel to the plane of the figure is the same size and form as the figure itself.

98. **To find the shadow of a square prism on the plane of the base.**

Let the prism be given as shown in Fig. 61.

Analysis. Since the shadow of the prism is to be cast upon the plane of the base, it is evident that the base of the prism coincides with its own shadow. The shadow of the top is found by passing rays of light through the points A, B, C, and D and finding where these rays pierce the plane of the base. The straight lines joining these piercing points, in the proper order, will form the outline of the shadow of the top. The shadows of the corners of the upper base joined with the shadows of the corresponding corners of the lower base will be the shadows of the lateral edges. This completes the outline of the shadow of the prism.

Construction. $b\ b_s$ is the top and $b'\ b'_s$ is the front view of a ray of light which passes through B. The front view b'_s of the point where this ray pierces the plane of the base is first found and then the top view b_s located. Then B_s is the shadow of the point B upon the plane of the base of the prism. In like manner, C_s, D_s, and A_s, are respectively the shadows cast by C, D, and A upon the plane of

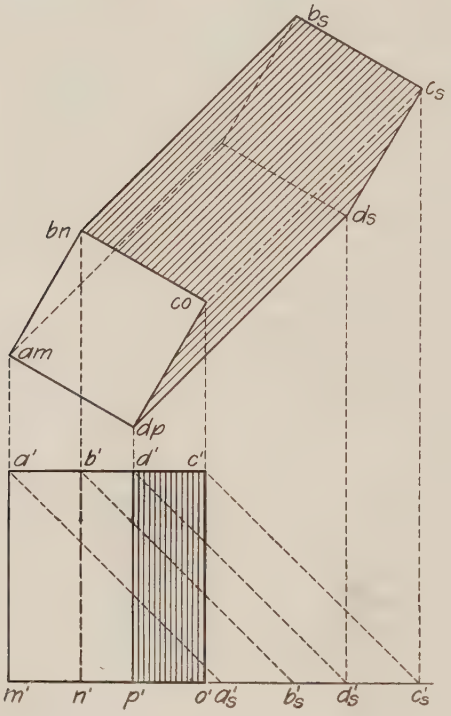


FIG. 61.—*Shadow of a square prism.*

the base of the prism. Joining these points in order gives the shadow of the top on the plane of the base. Joining B_s with N gives the shadow of the vertical edge BN. The shadows of the other vertical edges are found in the same manner. Then $nb_s\ c_s\ d_s\ p$ is the top view of the outline of the shadow of the prism on the plane of the base of the prism.

The face COPD of the object is turned away from the light and is therefore in the shade. This face is visible in the front view, and the fact that it is in the shade is indicated as shown in the figure.

From the figure, it is seen that a line and its shadow are parallel when the line is parallel to the plane upon which the shadow is cast.

99. To find the shadow of an object on the plane of the base and on the object itself.

Let the object be given as shown in Fig. 62.

This problem illustrates the finding of the shadow which an opaque object will cast upon itself as well as the plane of the base. It also shows the method of finding the shadow on a vertical plane in addition to finding the shadow on a horizontal plane.

Analysis. All of the edges shown are straight lines. If all of the shadow of any edge falls on one plane, it will be necessary to find the shadow of only two points of the edge and join them by a straight line. If the shadow of an edge falls on more than one plane, two shadow points should be found on each plane and joined by a straight line. The shadow of any point of an edge falls upon the plane which is first pierced by the ray of light passing through that point.

Construction. Each point of the top view of the shadow which the object casts upon the plane of the base is found as illustrated by points A and B, Fig. 61. For this shadow it is only necessary to find where the rays of light pierce the horizontal plane. The shadow which the upper part of the object casts upon the rectangular collar is found as illustrated by corner D, d'_s being first located then d_s.

The rectangular collar casts a shadow upon the vertical faces of the object. The horizontal edge BE of the collar

casts part of its shadow, b_s c_s, on the plane of the base and the remainder, c_s' o_s' e_s' on two of the vertical faces of the object. o_s and e_s are first located and then the front views o'_s and e'_s. To determine the part CO of the edge

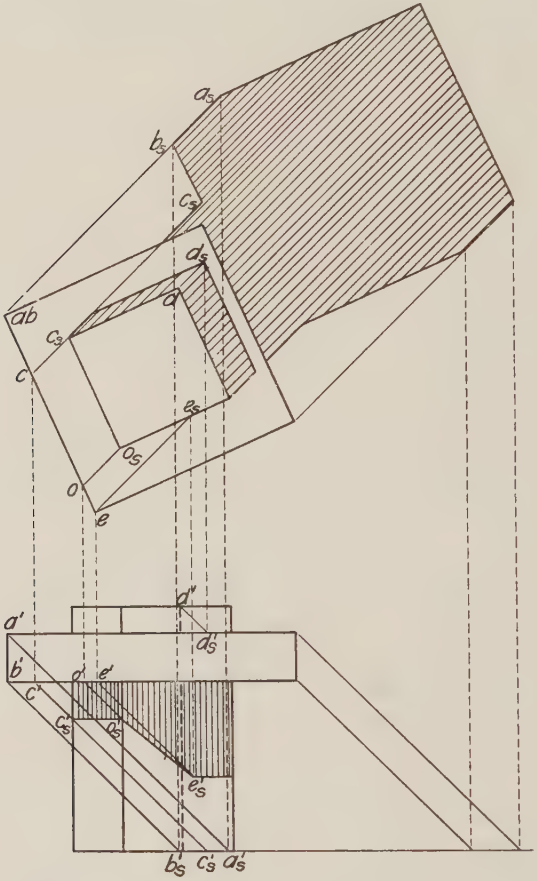


FIG. 62.—*Shadow of object on plane of base and object itself.*

BE which casts a shadow on the vertical face, draw the top views of rays of light through c_s and o_s, thus locating the points c and o. The front view shows the shadow which the collar casts upon the vertical faces of the object.

100. To find the shadow of a cylindrical column and cap on a horizontal plane and also the shadow of the cap on the column.

Let the column and the cap be given as shown in Fig. 63.

Analysis. Since the upper surface of the cap is a horizontal circle, its shadow on a horizontal plane is a

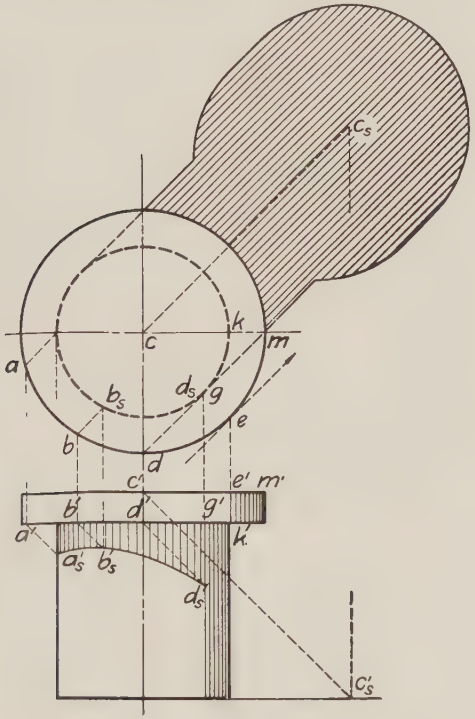


FIG. 63.—*Shadow of column and cap.*

circle of the same radius as the cap. The shadow of the center of the circle is the center of the shadow of the circle. Finding the shadow of the center and drawing a circle with a radius equal to the radius of the cap, will give the shadow of the top of the cap on a horizontal plane. The shadow of the lower surface of the cap on a horizontal

plane is found in the same way. Common tangents to these circles drawn in the direction of the top view of a ray of light are the shadows of the extreme elements of the cap and complete the shadow of the cap on the horizontal plane. The shadow of the column on a horizontal plane is bounded by two lines drawn in the direction of the top view of a ray of light and tangent to the base of the column.

To find the shadow of the cap on the column, pass rays of light through points of the lower edge of the cap and find where they pierce the column. The top view b_s, d_s, etc., of these points are first determined and subsequently, the front views b'_s, d'_s, etc. A line joining these points is the boundary of the shadow of the cap on the column.

The side of the column and cap turned away from the source of light is in the shade. GK is the portion of the column and EM the portion of the cap which are respectively in the shade and visible in the front view.

101. Problems.

Solve each of the following problems in a 7½″ x 11″ rectangle with the base line through the center and parallel to the shorter sides of the rectangle.

1. One edge of a 2″ cube lies on a horizontal plane and is oblique to V. The plane of one face of the cube makes 30° with H. Draw the top and front views of the cube and find its shadow on a horizontal plane.

2. Draw the top and front views of a 2″ cube when one diagonal is perpendicular to H. Find the shadow of the cube on a horizontal plane which passes through the lower end of the diagonal.

3. A right square pyramid has its vertex in a horizontal plane. The side of the base is 2″. The altitude is 3″ long, makes 60° with H, and is oblique to V. Draw the top and front views of the pyramid and find its shadow on a horizontal plane.

4. Draw the top and front views of a sphere which has a radius of 1¼″. Find at least twelve points in its shadow on a horizontal plane.

CHAPTER V.

CURVED LINES AND SURFACES

GENERATION AND CLASSIFICATION OF LINES

102. **A line** is the path of a moving point.

Lines are of two general classes:

I. **A straight line** or **right line** is the path of a point which moves always in the same direction.

II. **A curved line** is the path of a point which moves so as to change its direction continually.

Curved lines are of two kinds:

(A) **A curved line of single curvature** or **plane curve** is a curve in which all positions of the moving point lie in one plane. Examples, circle, ellipse.

(B) **A curved line of double curvature** or **space curve** is a curve in which four positions of the moving point do not lie in one plane, where these four points are taken one right after the other along the curve. Such points are frequently called "consecutive points." Example, the edge of an ordinary screw thread. The line of intersection of two curved surfaces is usually of this form.

OUTLINE

- Lines
  - I Straight or Right Lines
  - II Curved Lines
    - (A) Curves of Single Curvature or Plane Curves
      - Circle
      - Ellipse
      - Parabola
      - Hyperbola
      - Sinusoid
      - Cycloid
    - (B) Curves of Double Curvature or Space Curves
      - Helix

VIEWS OF A CURVE

103. The top view of a curve is the line joining the points in which the perpendiculars to H from points from the curve pierce a horizontal plane. Similarly, any other view is obtained by perpendiculars to the picture plane from points of the curve. Enough points must be determined in the views to give the desired degree of accuracy. If the form is known, the views of the curve can sometimes be constructed by determining only a few points, such as the view of a circle or ellipse.

Two views of a curve, on planes which are not parallel to each other, will usually determine the form of the curve in space. All views of a curve of double curvature are curved lines, but views of a plane curve from certain directions are straight lines. Answer the following questions regarding the views of a plane curve:

(a) In what position must the plane of the curve be placed in order that the top view of the curve be a straight line? In this position, how may the front view appear?

(b) How must the plane of the curve be placed in order to have the front view of the curve a straight line? Describe the top view of the curve.

(c) Can both top and front views of a curve be straight lines? Which view of the curve will show its true size?

(d) In general, what must be the direction of the line of sight to show a plane curve as a straight line? To show it in its true size?

TANGENTS AND NORMALS TO LINES

104. If in a secant line AB, Fig. 64, the point A be kept fixed and the point B moved along the curve until it coincides with A, **the secant AB becomes a tangent to the curve at the point A.**

Two curves are said to be tangent to each other at a point when they have a common tangent at that point.

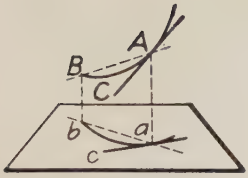


FIG. 64.—*Tangent to a curve.*

If a straight line is tangent to a plane curve, the tangent will lie in the plane of the curve. This is evident since the secant through the point of tangency is in the plane of the curve, and it remains in this plane as it moves to the position of the tangent.

105. If two lines are tangent to each other in space, their projections on the same plane will be tangent. Fig. 64 represents a curve in space with its secant AB. Let these lines be projected upon any plane. The projections of the points A and B will approach each other as the points A and B in space approach each other. When the secant AB becomes a tangent in space, the points A and B coincide and their projections will also coincide in the only point common to the projections of the straight line and curve. The projections or views of the lines are therefore tangent to each other at this common point.

If a straight line is tangent to a curve in space, its top view is tangent to the top view of the curve and its front view tangent to the front view of the curve. The top and front views of the point of tangency must lie in a line at right angles to the ground line.

106. Normals. If a line be drawn perpendicular to the tangent at the point of tangency, it is called a normal to the curve. There are an infinite number of normals to a curve at a point on the curve. However, the normal to a plane curve is understood to be the one lying in the plane of the curve unless otherwise stated.

107. Rectification of curves. To rectify a curve means to find a straight line equal in length to the curve. This is accomplished approximately by dividing the curve into a number of small arcs, so small that for all practical purposes the chords of these arcs may be taken as equal in length to the arcs themselves. The small chords are then laid off one after another along a straight line. This can be conveniently done with the dividers. The part of the line thus covered is the rectified curve.

As an exercise in the rectification of a curve, let the student draw a circle of two inch diameter and after rectifying the entire circumference by the above method, check the result by mathematical computation.

CURVES OF SINGLE CURVATURE

108. The most common plane curves are obtained by cutting a right circular cone by a plane. These curves are the circle, ellipse, parabola, and hyperbola. Fig. 65 shows the position which the cutting plane must take with reference to the plane of the base of the cone in order to cut the particular curve desired. ϕ must be less than θ to cut an ellipse from the cone and α greater than θ to cut an hyperbola.

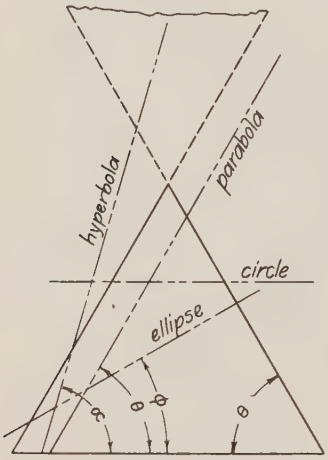


FIG. 65.—*Plane sections of a cone of revolution.*

A circle is the path of a point moving in a plane so that its distance from a given point is constant.

109. An ellipse is the path of a point moving in a plane so that the sum of its distances from two fixed points is constant.

Construction. Let F and F_1, Fig. 66, be the two fixed points and let the sum of the distances from these points to any point on the curve be equal to the line AB. Select any point, as O, on the line AB. With F as center and AO as radius, strike an arc; also with F_1 as center and BO as radius, strike an arc cutting the first arc in the points P and Q. These are points on the curve since $FP+F_1P=AB$ and $FQ+F_1Q=AB$. Selecting any other point on AB, as O_1, and going through the construction as before, two other points P_1 and Q_1 are located. By continuing this process enough points can be located so that the curve can be drawn. This is an accurate method for constructing the curve but in practice it leaves many unnecessary lines on the drawing.

The points F and F_1 are **foci** of the ellipse. AB is the **major axis.** The **minor axis** CD is perpendicular to the major axis at its middle point. The intersection of the axes is the **center** of the ellipse. Lines drawn from any point on the curve, as P, to the foci F and F_1 are called **focal radii.**

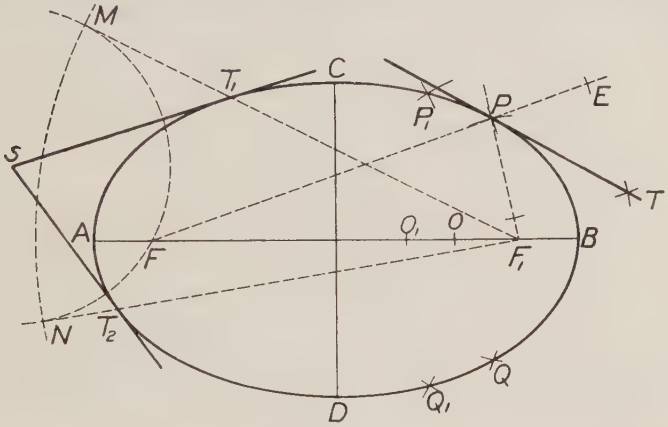


FIG. 66.—*Construction of an ellipse and tangents.*

If the axes of an ellipse are given, the foci can be found by taking one-half the major axis as a radius and with the end of the minor axis as a center, cut the major axis in two points which are the foci.

There are many other ways of constructing an ellipse. One of the best is that known as the "trammel method." This method is illustrated in Fig. 67, and is as follows: On a straight edge (the edge of a thin card or strip of paper is suitable for the purpose) lay off three points A, B, C, in such a way that AC is half the major and BC half the minor axis of the required ellipse. If the straight edge be moved in such a manner that the point A moves on the minor and B on the major axis, the point C will trace the ellipse. In case the major and minor axes are somewhat near the same length more accurate work can be obtained by laying off BC equal to one half the minor and BA one half the major axis. The point B, the middle point, will then trace the curve if A move on the minor and C on the major axis. In this case the axes must be extended beyond the circumference of the ellipse.

The most general statement of the trammel method is that any point in a plane will trace an ellipse if two other points of the plane are made to follow respectively two intersecting lines on the plane. The three points need not be in a straight line and the directing lines need not be at right angles to each other. In this case the major and minor axes of the ellipse will not fall along the directing lines.

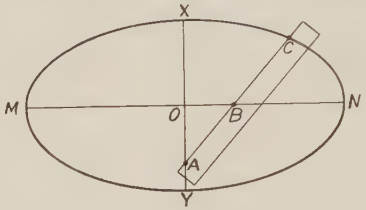


FIG. 67.—*Trammel method for constructing an ellipse.*

There are instruments, called ellipsographs, on the market which will trace an ellipse by the mere turning of a handle.

110. To draw a tangent to the ellipse at a point on the curve. Let P, Fig. 66, be the point on the curve. Draw the focal radii PF and PF_1. The line PT which bisects the angle F_1PE is the required tangent. (Wood's Co-ordinate Geometry, Art. 108A.) This method applies to all conics.

To draw a tangent to the ellipse from a point without the curve. Let S, Fig. 66, be the point without the curve. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points M and N. F_1M and F_1N intersect the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents. (Wood's Co-ordinate Geometry, Art. 108B.) This method applies to all conics.

The points M and N, at which the arcs intersect, must be joined to the focus used as a center when the major axis is used as a radius. Should the point from which the tangents are drawn be on the extension of the minor axis of the ellipse, the arc with this center passes through both foci. In this case either focus may be used as center for the arc having the major axis for a radius.

111. A parabola is the path of a point moving in a plane so that its distance from a given point is always equal to its distance from a given straight line.

Construction. Let F, Fig. 68, be the given point and AB the given straight line. Draw a line CD from F perpendicular to AB. Through any point, as O, on CD, draw a line parallel with AB. With F as center and CO as ra-

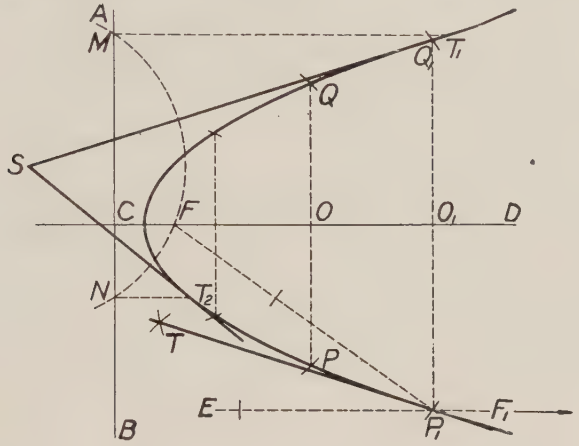


FIG. 68.—*Construction of a parabola and tangents.*

dius strike an arc cutting this line in two points P and Q. P and Q are points on the parabola, since FP and FQ are each equal to the distance of P and Q from AB. Selecting some other point on the axis and going through a similar construction, two other points of the curve are located. By continuing this process, enough points can be located so that the curve can be drawn.

The point F is the **focus**, the line AB the **directrix**, and the line CD the **axis** of the parabola.

112. To draw a tangent to the parabola at a point on the curve. Let P_1, Fig. 68, be the point on the curve. Draw the focal radii FP_1 and F_1P_1. (To make this construction similar to that for the tangent to the ellipse, it is necessary to consider one focus F_1 of the parabola at an infinite distance on the axis). The line P_1T which bisects the angle FP_1E is the required tangent.

Another accurate and convenient method for constructing a tangent to a parabola at a point on the curve comes from the fact that the sub-tangent is bisected by the vertex of the curve. For example, extend the axis CD, Fig. 68, to the left and lay off on it from the vertex the distance from the vertex to O_1. This point joined with P_1 gives the tangent TP_1.

The normal to the curve at the point P_1 bisects the angle FP_1F_1. Therefore, a ray of light from F striking the parabola at any point as P_1 will be reflected parallel to the axis CD. For this reason, reflectors are made in the form of a surface generated by revolving the parabola about its axis CD.

To draw a tangent to the parabola from a point without the curve. Let S, Fig. 68, be the point without the curve. With S as center and SF as radius, strike an arc cutting the directrix in the points M and N. MT_1, and NT_2, parallel with the axis CD of the parabola, cut the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents.

113. Problems.

Use a 5½″ x 7½″ rectangle for each of the following problems.

1. Construct the top and front views of an ellipse having A (—1, ½, —2½) B (3½, ½, —2½) for the major axis and C (1¼, 1¾, —2½) E (1¼, —¾, —2½) for the minor axis. Construct a tangent line to the curve at a point 3½″ from one of the foci. Construct a tangent line to the curve from a point 3½″ from the center.

2. Construct the top and front views of a parabola having F (0, ½, —2½) as focus and A (—¾, 1½, —2½) B (—¾, —½, —2½) as directrix. Construct a tangent line to the parabola at the point P (¼, ?, —2½). Construct a tangent line to the parabola from the point M (—2½, 0, —2½).

3. Construct the top and front views of a line tangent at the point C (—1½, ½, —1) to an ellipse having C for a curve point and A (—2½, 1½, —1½) and B (—½, 2¼, —2½) as foci. (The views of the ellipse are not required.)

4. A (—2, 1½, —1) is the end of the minor axis of an ellipse whose major axis is 4 inches long and lies along the line B (—2½, 0, —2¼) C (½, 1, —¾). Construct the top and front views of a line tangent to the ellipse at a point 2½ inches from a focus. (The views of the ellipse are not required.)

5. Construct the top and front views of a straight line which is tangent to the circle A (—2½, 1¼, —1½) B (—½, 2, —2½) C (—1½, ¼, —1) at the point B. The views of the circle are not required.

6. Construct the top and front views of one of the tangents from the point C (—1½, ¼, —1) to the circle which has A (—2½, 1¼, —1½) B (—½, 2, —2½) for a diameter. (The views of the circle are not required.)

7. Construct the top and front views of a line which is tangent to a circle A (—2½, ¾, —1½) B (—½, 1½, —2½) C (—1½, —¼, —1) and intersects the line O (0, 2½, —½) P (1, 0, —2½). (The views of the circle are not required.)

8. A (—2, 1½, —1) is the focus of a parabola whose directrix is along the line B (—2½, 0, —2½) C (½, 1½, —½). Construct the top and front views of a line tangent to the parabola at a point 1¼″ from the focus. (Views of the parabola not required.)

114. A hyperbola is the path of a point moving in a plane so that the difference of its distances from two fixed points is constant.

Construction. Let F and F_1, Fig. 69, be the two fixed points and let the difference of the distances from these points to any point on the curve be equal to the line AB.

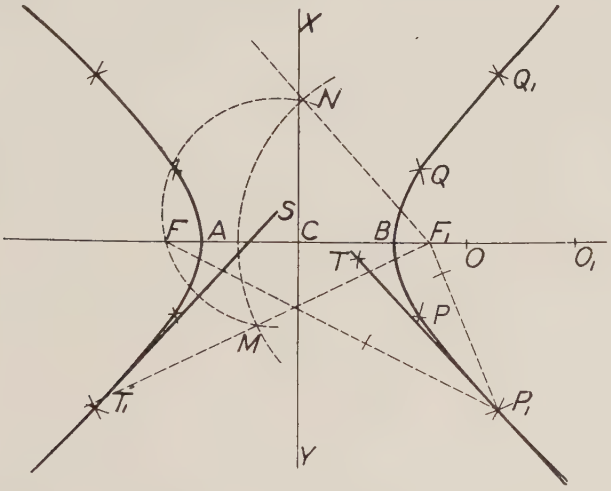


FIG. 69.—*Construction of the hyperbola and tangents.*

Select any point, as O, on the line AB extended. With F as center and AO as radius, strike an arc; also with F_1 as center and BO as radius, strike an arc cutting the first arc

in the points P and Q. These are points on the curve since $FP—F_1P=AB$ and $FQ—F_1Q=AB$. (If F_1 be taken as center and AO as radius and F as center and BO as radius, the points located will be on the other branch of the hyperbola). Selecting any other point on AB, as O_1, and going through the construction as before, two other points P_1 and Q_1 are located. By continuing this process enough points can be located so that the curves can be drawn.

The points F and F_1 are the **foci** of the hyperbola. The point C, midway between the foci F and F_1 is the **center** of the curve. The line AB is the **transverse axis.** The line XY, perpendicular to AB at its middle point, is the **indefinite conjugate axis** of the curve.

115. To draw a tangent to the hyperbola at a point on the curve. Let P_1, Fig. 69, be the point on the curve. Draw the focal radii FP_1 and F_1P_1. The line P_1T which bisects the angle F_1P_1F is the required tangent.

To draw a tangent to the hyperbola from a point without the curve. Let S, Fig. 69, be the point without the curve. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points M and N. F_1M and F_1N intersect the curve in the points of tangency. ST_1 is one of the tangents. The other point of tangency, where NF_1 intersects the curve, is without the limits of the drawing.

CURVES OF DOUBLE CURVATURE

116. An ordinary helix is the path of a point moving on the surface of the cylinder of revolution so as to intersect its elements at a constant acute angle.

The axis of the cylinder is the **axis** of the helix.

Construction. Let MN, Fig. 70, be the axis of the helix and P the generating point.

Suppose that for one complete turn around the axis the generating point moves through a vertical distance $m'\ n'$. This distance is the pitch of the helix.

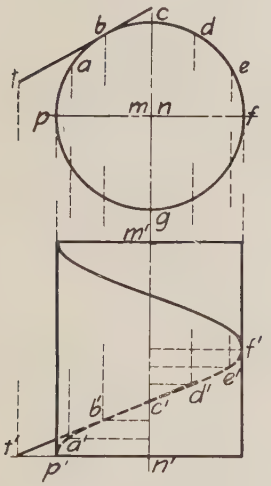


FIG. 70.—*Construction of helix and tangent.*

Since the curve is on the surface of a cylinder of revolution a view of it looking in the direction of the axis will be the circle *pcfg*.

To draw the view looking at right angles to the axis of the helix, in this case the front view, divide the circle *pcfg* into any number of equal parts, as twelve, and $m'\ n'$ into the same number of equal parts. Draw horizontal lines through the points of division of $m'\ n'$. Since the motions of the point around and along the axis are both uniform, the point in making one-twelfth of a complete turn around the axis will rise one-twelfth of the distance $m'\ n'$. a is the top and a' the front view of the point after making one-twelfth of a turn. In the same manner, the points b', c', d', etc., are found. The front view of the curve is drawn through the front views of these points. The view of the helix at right angles to the axis is the sinusoid.

Since the curve cuts all the elements of the cylinder at the same angle, it is evident that the helix will become a straight line if the cylinder is opened along an element and the surface rolled out into a plane. From this position it

is seen that the hypothenuse of a right triangle will form a helix if the altitude of the triangle coincides with an element of a right circular cylinder and the base of the triangle is wound around the base of the cylinder.

117. To construct a tangent to a helix at a point on the curve. Let B, Fig. 70, be the point at which the tangent is to be drawn. Let the triangle referred to in the above paragraph as being wound around the cylinder, be unrolled as far as the point of tangency B. The hypothenuse of the triangle in this position is tangent to the curve, since it touches the curve at the point B and makes the same angle with the horizontal plane as the curve. The base *bt* of the triangle is the top view of the tangent, and is equal in length to the arc *bp* of the circle *pcfg*. Therefore, to get the views of the tangent, draw its top view *bt* tangent to the circle *pcfg* at the point *b*. On this line lay off from *b* the true length of the arc *bap* of the circle. This will give the point *t* where the tangent pierces the plane of the base. The front view of this piercing point is at *t'* and this joined with *b'* gives the front view *b' t'* of the tangent.

118. Problems.

Solve each of the following problems in a 5½" x 7½" rectangle with the base line through the center and parallel to the longer sides of the rectangle.

1. Construct the top and front views of the helix traced by the point O (—1, 1¼, —¼) as it moves about A (0, 1¼, —¼) B (0, 1¼, —2½) as an axis. Pitch 2¼". Construct a tangent to the helix at a point ¾" higher than B.

2. Construct the front and left end views of one turn of a helix whose axis is parallel to G. L. Diameter of helix 2½", pitch 3". Construct a tangent line to the helix at some point on the curve.

3. Construct the top and front views of a hyperbola lying in a horizontal plane. F (0, —¾, —2¼) and F_1 (0, ¾, —2¼) are the foci, and the transverse axis is 1" in length. Construct a tangent line to the curve at a point 1" from a focus. Construct a tangent line to the curve from the point S (—¼, ½, —2¼).

4. C (—1¼, ½, —¾) is a curve point of a hyperbola whose foci are A (—2½, 1, —¾) and B (—½, 2, —2¼). Construct a tangent line to the hyperbola at the point C. (The views of the hyperbola are not required).

GENERATION AND CLASSIFICATION OF SURFACES

119. A surface is the path of a moving line. An exception to this is the case of a straight line moving in the direction of its length.

The moving line is the **generatrix**, and its different positions are the **elements** of the surface.

Surfaces are of two general classes, ruled and double curved.

I. A ruled surface is one which can be generated by the motion of a straight line. A straight edge can therefore be placed against the surface in such a manner that it touches the surface throughout its length. Examples, cylinder, cone, helicoid.

II. A double curved surface is one which can only be generated by the motion of a curved line. Such a surface has no straight line elements. Examples, sphere, ellipsoid.

Ruled surfaces are of three kinds, plane, developable, and warped.

(A). A plane surface, or a **plane**, is a surface in which, if any two points are taken, the straight line joining these points lies wholly in the surface.

(B). A developable surface is a ruled surface in which any two straight line elements, coming one right after the other on the surface, lie in a plane. The surface can therefore be rolled into a plane or developed without undergoing distortion. Examples, cylinder, cone.

(C). A warped surface is a ruled surface in which any two straight line elements, coming one right after the other on the surface, do not lie in a plane. The surface cannot be developed without undergoing distortion. Example, the surface of a screw thread.

Developable surfaces are of three kinds, cylinders, cones, and convolutes.

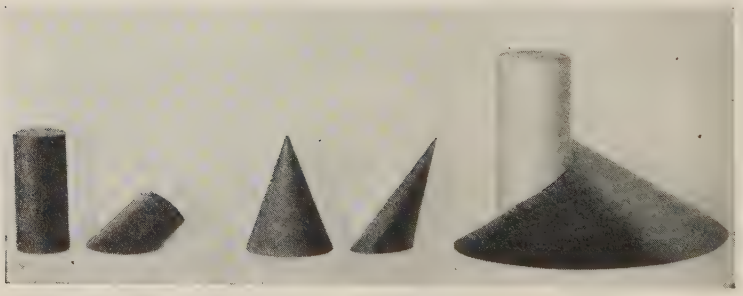
I. A cylinder is a developable surface in which the rectilinear elements are parallel.

II. A cone is a developable surface in which the rectilinear elements intersect in a point.

III. A convolute is a developable surface in which the rectilinear elements are tangent to a curve of double curvature. Example, the surface having tangents to a helix for its elements.

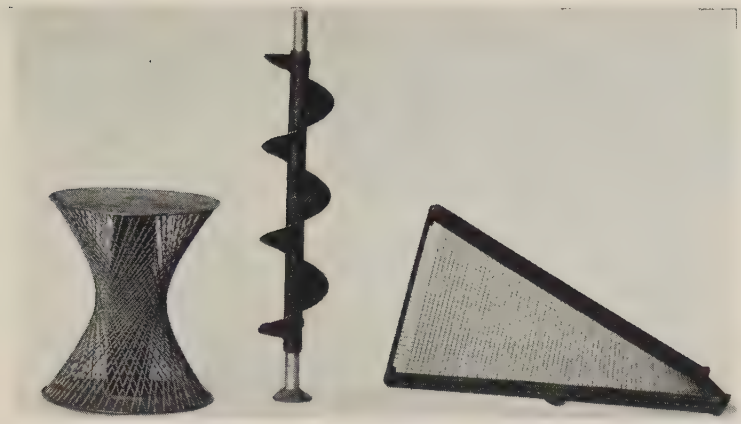
OUTLINE OF SURFACES

- Surfaces
  - I Ruled
    - (A) Plane
    - (B) Developable
      - I Cylinder
      - II Cone
      - III Convolute
    - (C) Warped
      - Helicoid
      - Hyperbolic Paraboloid
      - Conoid
      - Cylindroid
      - Hyperboloid of Revolution of one Sheet
      - ------etc.
  - II Double Curved
    - Sphere
    - Torus or Anchor Ring
    - Ellipsoid of Revolution.
    - Paraboloid of Revolution.
    - Hyperboloid of Revolution of two Sheets
    - ------etc.



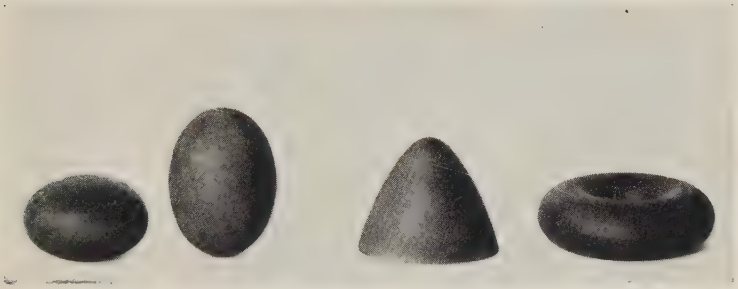
Cylinders *Cones* *Convolute*

FIG. 71.—*Developable surfaces.*



Hyperboloid of Revolution of one Sheet. *Helicoid.* *Hyperbolic Paraboloid*

FIG. 72.—*Warped surfaces.*



Oblate Ellipsoid *Prolate Ellipsoid* *Paraboloid* *Torus or Anchor Ring*

FIG. 73.—*Double curved surfaces.*

SURFACES OF REVOLUTION

120. A surface of revolution is the path of a line *revolving* about a straight line as an axis.

It is evident that the intersection of this surface with a plane perpendicular to the axis is a circle. A surface of revolution may be generated by a circle having its center moving along a straight line, its different positions in parallel planes, and its radius changing according to a given law.

The intersection of the surface with a plane containing the axis is a **meridian line,** and the plane is a **meridian plane.** All meridian lines of the same surface are identical. Any surface of revolution may be generated by revolving its meridian line about its axis.

OUTLINE OF SURFACES OF REVOLUTION

- Surfaces of Revolution
  - Ruled
    - Developable
      - Right circular cylinder
      - Right circular cone
    - Warped
      - Hyperboloid of Revolution of one Sheet
  - Double curved
    - Sphere
    - Torus
    - Ellipsoid of Revolution
      - Prolate
      - Oblate
    - Paraboloid of Revolution
    - Hyperboloid of Revolution of two sheets
    - .....etc.

121. If two surfaces of revolution have a common axis, what will be their line of intersection?

What must be the relative position of the axis and the generatrix to form a cylinder of revolution? A cone of revolution? A hyperboloid of revolution of one sheet?

The cylinder and cone are the only developable surfaces of revolution.

The hyperboloid of revolution of one sheet is the only warped surface of revolution.

TANGENT PLANES TO SURFACES, NORMAL LINES AND PLANES

122. Let AD_1 and AD_2, Fig. 74, be any two intersecting curves on a surface, and BCD a curve which if moved along the curves AD_1 and AD_2 will generate the given surface. The curve BCD may vary in form as it moves. The secants AB, BC, and AC lie in one plane. As the curve BCD moves toward A, the points of intersection B and C will travel along the curves AD_1 and AD_2 until the secants AB and

AC become the tangents AB_1 and AC_2 at the point A. The curve AD_3 is the position of the moving curve BCD when the points B and C reach A. In this position, the secant BC becomes the tangent B_3C_3. The tangents AB_1, AC_2, and B_3C_3 all lie in one plane which is called the **tangent plane** to the surface at the Point A. The point A is the **point of contact.**

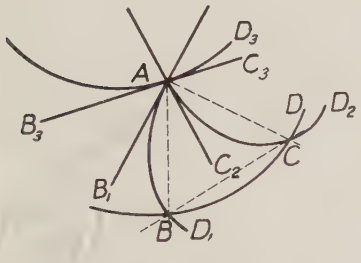


FIG. 74.—*Tangent plane to a surface.*

Since AD_1 and AD_2 are any curves of the surface, it follows that in general, the tangent at A to every curve of the surface through this point will lie in the tangent plane. Therefore, **the tangent plane will contain all straight lines tangent to lines of the surface at the point of contact.**

If any plane be passed through the point of contact, it will cut a straight line from the tangent plane and a line from the surface, and these lines will be tangent to each other.

Since two intersecting lines will determine a plane, it follows that **a plane tangent to a surface at a given point is determined by two straight lines tangent at this point to two lines of the surface.**

In any ruled surface, the rectilinear element through the point of contact lies in the tangent plane.

123. **A tangent plane to a developable surface is tangent all along a rectilinear element.** In Fig. 75 let BD be the curve of the base of any developable surface as a cylinder and AB and A_1B_1 any two rectilinear elements of the surface. From any point A of one element draw on the surface a curve AD_1 which cuts the other element at A_1. The chords AA_1 and BB_1 lie in the same plane and will intersect at some point as S_1. Now if the plane ABB_1A_1 be revolved about AB as an axis, the point A_1 will approach A along the curve AD_1 and B_1 will approach B along the curve

BD. When the point A_1 reaches A, the secant AS_1 becomes the tangent AS to the curve AD_1, and at the same time B_1 reaches B, the secant S_1B becoming the tangent SB to the curve BD. The plane SAB is tangent to the cylinder at the point B since it contains the tangent SB to the curve BD, and also the rectilinear element AB. This plane is also tangent to the cylinder at A, containing the tangent SA to the curve AD_1 and the element AB. Since A was taken as any point on the element AB, the plane SAB must be tangent to the surface at every point of this element.

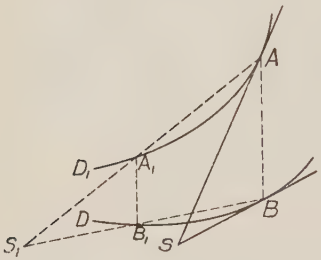


FIG. 75.—*Tangent plane to a developable surface.*

The same demonstration applies to the tangent plane to a cone.

In the case of the convolute, the element A_1B_1 is a curve instead of a straight line. The curve will change its form as the secant plane revolves, finally becoming a straight line coinciding with the axis AB. Otherwise the demonstration given above also applies to the convolute.

Since the plane is tangent all along an element of the surface, the intersection of the tangent plane with the plane of the base of the surface will be a straight line tangent to the curve which represents the base of the surface.

124. Tangent planes to warped surfaces. Since warped surfaces have straight line elements, the tangent plane to the surface at any point will contain the rectilinear element through that point. If any curve of the surface be drawn through the point of contact, the straight line tangent to the curve at this point and the rectilinear element through the point will determine the tangent plane to the surface.

If the warped surface is of such a form that there are two rectilinear elements through each point of the surface, then the tangent plane will be determined by the two elements through the point of contact.

Although the tangent plane to a warped surface contains a rectilinear element of the surface, it is, in general, tangent at only one point of this element. The tangent plane usually intersects the surface. These principles will be better understood when the tangent planes to some of the surfaces are drawn.

125. Tangent planes to double curved surfaces. Through the point on the surface at which the tangent plane is to be passed, draw any two curves of the surface. The straight lines tangent to these curves at their point of intersection will determine the tangent plane. If care is taken to select simple curves of the surface in simple positions with reference to the planes of projection, the tangents to these can be easily drawn.

In the case of a **double curved surface of revolution**, the simplest curves are usually the meridian curve and a circle which lies in a plane perpendicular to the axis of the surface.

126. Normals. A straight line perpendicular to the tangent plane at the point of contact is the **normal to the surface** at that point. Any plane containing a normal line is a **normal plane** to the surface.

DEVELOPABLE SURFACES

CYLINDERS

127. A cylinder is the path of a straight line which, as it moves, touches a curved line and has all its positions parallel.

The curved line is the **directrix** and the moving line is the **rectilinear generatrix.**

The directrix can be a closed curve, as the circle or ellipse, or it can be a curve which is not closed, as the hyperbola or helix.

The intersection of the cylinder with a horizontal plane is usually taken as the **base,** although any plane section may be considered to be the base. If this base has a center, the straight line through it and parallel to the rectilinear elements is the **axis of the cylinder.**

A right cylinder is one in which the rectilinear elements are perpendicular to the plane of the base. **A right circular cylinder** or **cylinder of revolution** is one formed by revolving a straight line about an axis to which it is parallel.

A section taken perpendicular to the elements of any cylinder is a **right section.** Cylinders are distinguished by the names of their right section; as circular cylinders, elliptical cylinders, etc. If a cylinder is intersected by a plane parallel to the rectilinear generatrix it will cut rectilinear elements from the surface.

128. To represent the surface. A cylinder is usually represented by the views of its base and its extreme elements.

In Fig. 76, AECG is the base and EF the rectilinear generatrix. The extreme elements in the top view are tangent to the base and parallel to the top view of the rectilinear generatrix. The extreme elements in the front view are drawn through the extreme points of the front view of the base and parallel to the front view of the rectilinear generatrix. It will be noted in general that the extreme elements which are tangent to the base in the top view are not the extreme or "limiting" elements in the front view.

Since the surface is indefinite in extent, it can be represented with the upper end broken off unless some definite portion of the surface is taken for a particular purpose.

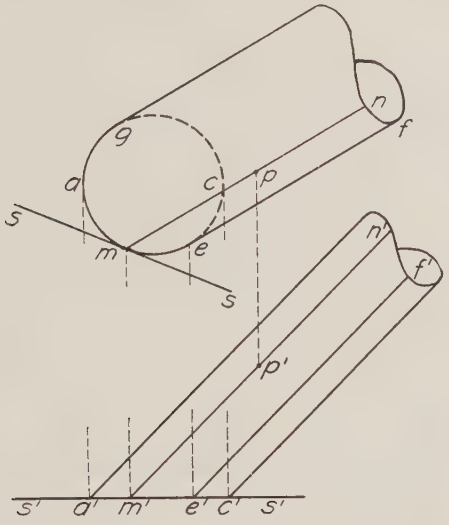


FIG. 76.—*Oblique cylinder and tangent plane.*

The line representing the conventional break at the end of the cylinder, Fig. 76, is drawn free hand. The top and front views do not represent the same curve on the cylinder.

129. To represent a rectilinear element of the surface. Through any point M of the base draw a straight line MN parallel to the rectilinear generatrix; this will be an element of the surface.

To represent a point of the surface. Locate a rectilinear element, as MN, and then take any point P of this element, Fig. 76. If one view of a point P on the surface of the cylinder is given, the other view is found in the following manner. Through the given view of P, draw the view of an element and extend it until it cuts the curve of the base of the cylinder. By means of this point on the base determine the other view of the element and then locate the other view of P on this view of the element.

130. To represent a plane which contains a given point and is tangent to a cylinder.

Case 1. When the given point is on the surface of the cylinder.

Let the cylinder be given as in Fig. 76, and let P be the given point.

Analysis. The tangent plane to a cylinder is tangent all along a rectilinear element. Consequently the line of intersection of the tangent plane with the plane of the base of the cylinder is tangent to the curve of the base (Art. 123, last paragraph). The tangent plane to the cylinder at the point P will be represented by the element through P and a straight line tangent to the curve of the base of the cylinder at the point where the element through P meets the base.

Construction. Draw the top and front views of the element MN which contains the point P. At M draw a tangent to the base of the cylinder. ss is its top and $s's'$ its front view. Then MNSS represents the tangent plane to the cylinder at P.

Case 2. When the given point is not on the surface of the cylinder.

Let the cylinder be given as in Fig. 77, and let P be the given point.

Analysis. A straight line through the given point and parallel to the rectilinear elements of the cylinder will lie in the required tangent plane. From the point in which this line pierces the plane of the base, draw a tangent to the curve of the base of the cylinder. This tangent and the line through the given point parallel to the elements represent the required tangent plane. From the point of tangency on the curve of the base, the element of contact can be drawn. This element also lies in the tangent plane.

Construction. Since the given cylinder is right circular and the axis AB is oblique to both H and V, it is best to solve the problem by the two successive auxiliary view method. The first auxiliary view shows the axis AB in its true length, and the point P at p_1. The second auxiliary view is taken in the direction of AB and shows the axis as

the point a_2b_2 and shows the point P at p_2. With a_2b_2 as center and the proper radius, draw the circle which represents the cylinder. The tangent p_2o_2 to the circle represents the required tangent plane. m_1 is the auxiliary view

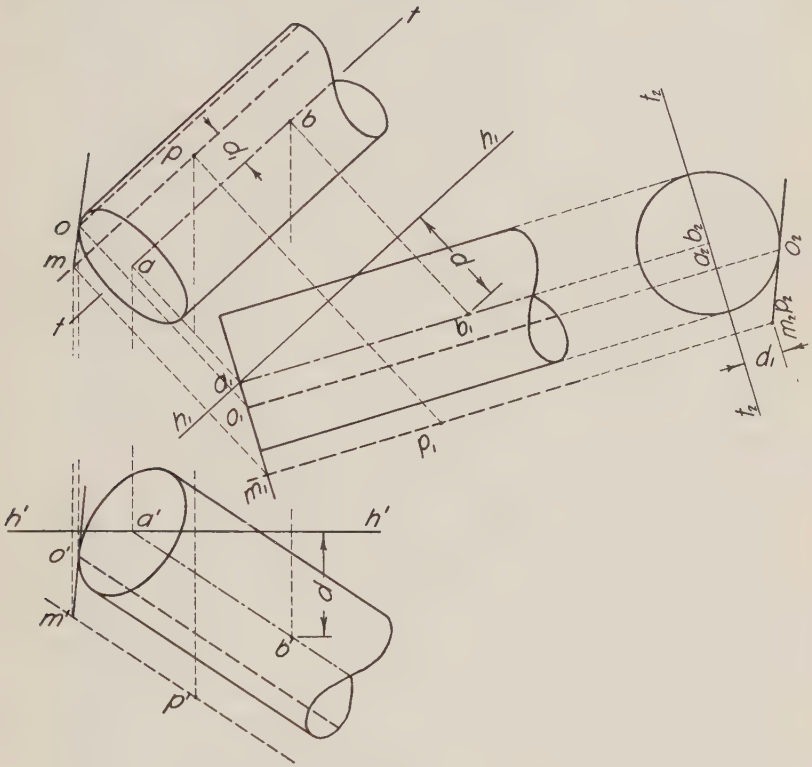


FIG. 77.—*Plane containing point P and tangent to cylinder.*

of the point in which the line through P parallel to the elements of the cylinder pierces the plane of the base. From the auxiliary view, the top view m and the front view m' are located. mo is tangent to the ellipse representing the base of the cylinder in the top view and $m'o'$ tangent to the ellipse representing the base in the front view. MPO is the required tangent plane. The element of contact which also lies in the tangent plane is drawn from O parallel with the axis AB.

There are two solutions to the problem. The other tangent plane to the cylinder is not represented in Fig. 77.

If the plane of the base of the cylinder had been perpendicular to either H or V so that the point M and the curve of the base could have been located easily, it would not have been necessary to take an auxiliary view to determine the required tangent plane.

131. Problems.

Solve each of the following problems in a 7½" x 11" rectangle with the base line through the center and parallel to the shorter sides of the rectangle.

1. (a) A right circular cylinder, axis A (0, 1½, —2¾) B (2, 3½, —2¾), diameter 2½" and a point P (—¾, 3¼, —2¼) are given by their top and front views. Represent a plane which contains the point and is tangent to the cylinder.

(b) Same as above, except with axis of cylinder C (0, 2, —2½) E (2½, 2, —2½) and point M (1, 3¾, —1¾).

2. Given a point P (—¾, 1½, —1¼) and a right circular cylinder with axis A (—2⅞, 1¾, —2½) B (⅛, 1¾, —2½). The diameter of the cylinder is 1½". Represent a plane which contains the point and is tangent to the cylinder.

3. Represent a plane which makes 45° with H and is tangent to a right circular cylinder with axis A (—2, 3, —2½) B (⅜, 1⅝, —2½). The diameter of the cylinder is 2".

4. A (—3¼, 1, —2) B (—½, 3, —4) is the axis of an oblique cylinder with circular bases (2" diameter) in profile planes. Construct a plane which is tangent to the cylinder at the point P (—1½, ?, —2½).

5. Represent a plane which is parallel to a given straight line A (0, 1½, —2) B (2, ¾, —2½) and is tangent to a cylinder which has O (—2, 1½, —4) X (¾, 2⅜, —⅜) for an axis. The base of the cylinder is a circle (2" diameter) in H. (A plane containing the given line and a line parallel to the elements of the cylinder will be parallel to the required tangent plane.)

6. X (—2½, 1½, —2) Y (0, 4, —4½) is the axis of a right circular cylinder (2" diameter). Draw the top and front views of a plane which contains the point P (—2, 3, —4½) and is tangent to the cylinder. (The top and front views of the cylinder are not required.)

7. X (0, 3, —2) Y (—2, 0, —5) is the axis of an oblique cylinder whose base is a horizontal circle (2½" diameter) with center at Y. Construct a plane which is tangent to the cylinder and makes 60° with H. How many solutions are there to this problem? (A plane which contains the axis of the cylinder and makes 60° with H will be parallel to the required tangent plane. See Art. 70.

8. A (—1½, 1, —¾) B (¾, 4¼, —3½) is the axis of a cylinder whose base is a 2½" circle in V. Construct a plane which is tangent to the cylinder and makes 60° with V. How many solutions are there to this problem? (A plane which contains the axis of the cylinder and makes 60° with V will be parallel to the required tangent plane. See Art. 70.)

CONES

132. A cone is the path of a straight line which, as it moves, touches a curved line while one point of the straight line remains fixed. If the directrix is a plane curve, the fixed point does not lie in the plane of the curve.

The curve is the **directrix,** the fixed point the **vertex,** and the moving line the **rectilinear generatrix** of the cone.

The directrix can be a closed curve, as a circle or an ellipse, or it can be a curve which is not closed, as a hyperbola or a helix.

The generatrix being indefinite in length, will generate a surface of two parts which are on opposite sides of the vertex and which are called the **upper and lower sheets.**

The intersection of the cone with a horizontal plane is usually taken as the **base,** but any plane section of the cone may be considered the base. Cones are distinguished by the names of their bases; as circular cones, elliptical cones, etc.

A right cone or **a cone of revolution** is one in which all the rectilinear elements make the same angle with a straight line through the vertex called the **axis** of the cone. In the oblique cone, the line which joins the vertex with the center of the curve taken for the base will be considered the axis.

Any plane passing through the vertex and intersecting the cone will cut rectilinear elements from the surface.

133. To represent the surface. A cone is usually represented by the views of its base, vertex, and extreme elements.

In Fig. 78, AECG is the base and V the vertex of the cone. The extreme elements in the top view are tangent to the base and pass through the top view of the vertex. The extreme elements in the front view are drawn through the extreme points of the front view of the base and through the front view of the vertex. It will be noted that the elements tangent to the base in the top view are not the same as the extreme or limiting elements in the front view.

134. To represent a rectilinear element of the surface. Through any point M of the base, draw a straight line through the vertex V; this will be an element of the surface.

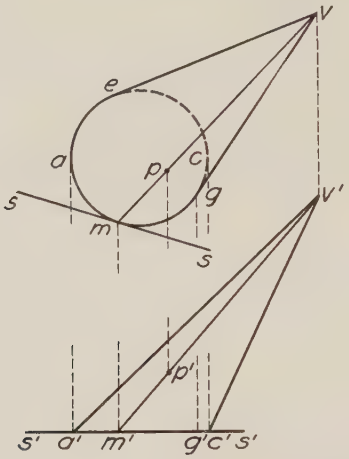


FIG. 78.—*Oblique cone and tangent plane.*

To represent a point of the surface. Locate a rectilinear element, as MV, and take any point P of this element. If one view of a point P on the surface of a cone is given, the other view is found in the following manner. Through the given view of P draw the view of an element from the vertex to the curve of the base of the cone. Join the other view of this point on the curve of the base with the other view of the vertex and then locate the other view of P on this view of the element of the cone.

135. To represent a plane which contains a given point and is tangent to a cone.

Case 1. When the given point is on the surface of the cone.

Let the cone be given as in Fig. 78, and let P be the given point.

Analysis. The tangent plane to a cone is tangent all along a rectilinear element. Consequently the line of intersection of the tangent plane with the plane of the base

of the cone is tangent to the curve of the base (Art. 123, last paragraph). The tangent plane to the cone at the point P will be represented by the element through P and a straight line tangent to the curve of the base of the cone at the point where the element through P meets the base.

Construction. Draw the top and front views of the element VM which contains the point P. At M draw a tangent to the base of the cone. ss is its top and $s's'$ its front view. Then VMSS is the tangent plane to the cone at the point P.

Case 2. When the given point is not on the surface of the cone.

Let the cone be given as in Fig. 79, and let P be the given point.

Analysis. A straight line joining point P with the vertex of the cone will lie in the required tangent plane. From the point in which this line pierces the plane of the base of the cone, draw a tangent to the curve of the base. This tangent and the line joining P with the vertex will represent the required tangent plane to the cone. The line joining the point of tangency on the base to the vertex is the element of contact. This element also lies in the tangent plane.

Construction. Since the given cone is right circular and the axis AB is oblique to both H and V, it is best to solve the problem by the two successive auxiliary view method. The first auxiliary view shows AB in its true length a_1b_1, and the point P at p_1. The second auxiliary view is taken in the direction of AB and shows the axis as the point a_2b_2. With a_2b_2 as center and the given radius, draw the circle of the base of the cone. The line a_1p_1 pierces the plane of the base of the cone at m_1. m_2 is the second auxiliary view of this point. From m_2 draw a tangent to the circle of the base, thus locating the point of tangency o_2. From o_2, the points o_1, o, and o' can be located. Joining O with the vertex A gives the element of contact. Then AOM represents the required tangent plane.

There are two solutions to the problem. The other tangent plane is not shown in Fig. 79.

The minor axis of the ellipse representing the base of the cone in the top view is obtained by projection from the first auxiliary view. Having the major and minor axes, the el-

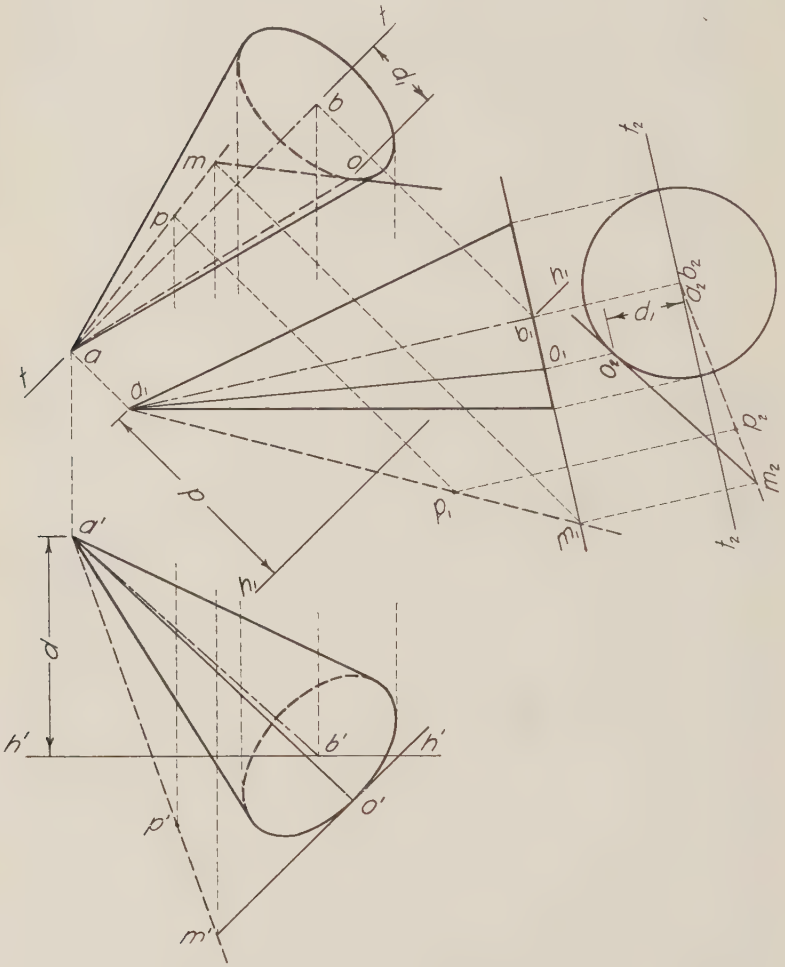


FIG. 79.—*Plane containing point P and tangent to cone.*

lipse can be drawn by the trammel method. By means of an auxiliary view from the front view, showing the axis AB true length, the length of the minor axis of the ellipse

in the front view is found. Then the ellipse representing the front view of the base can be drawn by the trammel method.

If the plane of the base of the cone had been perpendicular to either H or V so that the point M and the curve of the base could have been located easily, it would not have been necessary to take an auxiliary view to determine the tangent plane to the cone.

136. Problems.

Solve each of the following problems in a 7½" x 11" rectangle with the base line through the center and parallel to the shorter sides of the rectangle.

1. A (0, 2, —½) is the vertex of a right cone whose base is a circle (2½" diameter) in H with center at B (0, 2, —3½). Represent a plane which contains the point P (1¾, 2¾, —2⅞) and is tangent to the cone.

2. A (—2¾, 1½, —2½) B (¼, 1½, —2½) is the axis of a right cone whose base is a circle (2" diameter) in a profile plane. Represent a plane which contains the point P (—1¼, 2½, —2¼) and is tangent to the cone.

3. Given a point P (—1, 3½, —3) and a right cone which has A (—3, 3, —2½) B (—½, 1½, —2½) for an axis. The base is a circle 3" in diameter with center at B. Represent a plane which contains the point and is tangent to the cone.

4. Given a cone whose base is a circle (2½" diameter) in H with center at O (—1¼, 2, —3½). The vertex is at V (1¾, 3¾, —½). Represent a plane which is tangent to the cone at the point P (⅛, 2⅛, ?) on the surface.

5. Represent a plane which is parallel to the line M (0, 1½, —¼) N (2, 1, —3¾) and is tangent to the cone whose base is a circle (2½" diameter) in H with center at O (—2, 1½, —4¼), and whose vertex is at V (1, 4, —¼). (A line through the vertex of the cone and parallel to the given line will lie in the required tangent plane.)

6. A (—3, 1½, —2) B (—½, 3¼, —4) is the axis of a right circular cone. Base of cone is 2 inches in diameter with center at B. Represent the top and front views of a plane which contains the point P (—1, 3½, —2¼) and is tangent to the cone. (The top and front views of the cone are not required.)

7. V (0, 2¾, —2) is the vertex of an oblique cone whose base is a horizontal circle (2½" diameter) with center at A (—2, 0, —5). Represent a plane which makes 60° with H and is tangent to the cone. How many solutions are there to this problem? (See analysis, Article 70).

8. A cone whose base is a 2" circle in V, has A (—¼, ¾, —3) B (1¼, 4¼, —¾) for an axis. Construct a plane which is tangent to the cone and makes 60° with V. How many solutions are there to this problem? (See analysis, Art. 70).

CONVOLUTES

137. A convolute is the path of a straight line which, as it moves, is always tangent to a curve of double curvature such as the helix. The surface, Fig. 80, is represented by drawing the views of the curvilinear directrix, an occasional element, and the intersection of the surface with a horizontal plane. If the curvilinear directrix is a helix with axis perpendicular to the horizontal plane, the base of the surface will be the involute of the circle which represents the top view of the helix. When the directrix is a helix, the surface is also called a **helical convolute** or a **developable helicoid.**

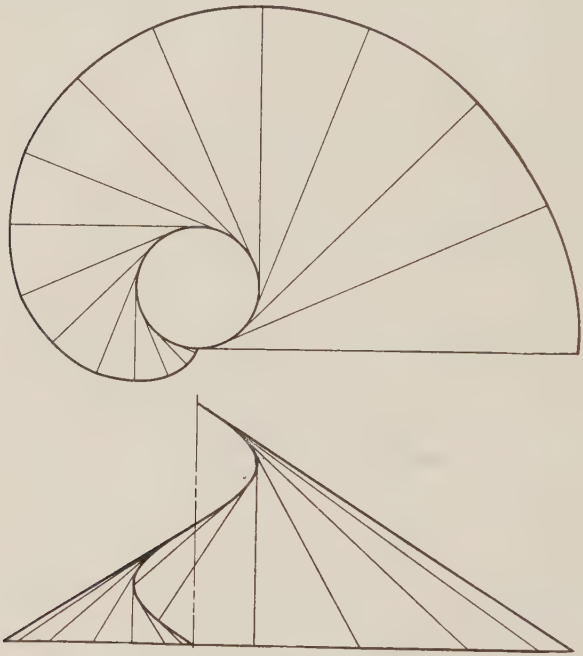


FIG. 80.—*Helical convolute.*

This surface is of very little practical importance although it has been used in the construction of screw conveyors. No further discussion of it will be given. For further information on the convolute, see Salmon's Geometry of Three Dimensions, Art. 349.

WARPED SURFACES

138. There is a great variety of warped surfaces, differing from each other in their form and manner of generation. Only a few of the more common ones will be discussed.

There are several kinds of warped surfaces formed by moving a straight line to touch two other lines, straight or curved, and keeping it parallel to a given plane, called a **plane directer.**

Warped surfaces are usually represented by drawing the views of one or more curves of the surface and a few of the rectilinear elements. The views of the directrices are often given.

HYPERBOLOIDS OF REVOLUTION OF ONE SHEET

139. **A hyperboloid of revolution of one sheet** is the path of a straight line revolving about another straight line as an axis, the generatrix and axis not being in the same plane.

From the nature of the motion it is evident that no two successive positions of the generatrix lie in one plane, therefore the surface is warped.

This is the only warped surface of revolution.

Upon this surface, skew bevel gears can be cut. Such gears can be used on a pair of non-parallel non-intersetting shafts. This form of gearing however is rarely used. One or more pairs of hyperboloids are used in machines for straightening iron pipes or rods.

140. **To represent the surface.** Let the surface be formed by the rotation of line CE about the axis AB, Fig. 81. Each point of CE as P will move in the circumference of a circle with center on AB and plane perpendicular to

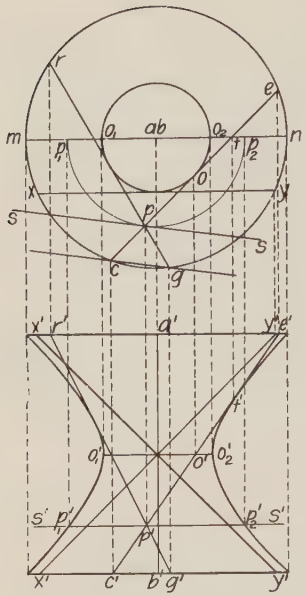


FIG. 81.—*Hyperboloid of revolution of one sheet.*

AB. The top view of this path is the circle $p\ p_1\ p_2$ and its front view is the straight line $p_1'\ p_2'$. The points P_1 and P_2 are in the meridian section MN parallel with V and are therefore on the curve representing the surface in the front view. By selecting other points on the generating line CE and revolving them into the plane MN about AB as an axis, other points on the curve representing the surface in the front view can be found. The hyperbola drawn through these points represents the surface in the front view. If this hyperbola be revolved about AB as an axis it will generate the surface of the hyperboloid. In the top view, a perpendicular to *ce* from the point *ab* locates the point *o*. The point O in space is the point of CE which is nearest the axis AB, and will therefore generate the smallest circle of the surface. This is called the **circle of the gorge.** The circle formed by any other point of the line CE as C may be considered the **base of the surface**. The top view of the base and the circle of the gorge is the top view of the surface.

The generating line CE pierces the meridian plane MN at the point T. $c'\ e'$ is tangent to the hyperbola representing the surface in the front view at t'. In a similar manner the front views of all rectilinear elements of the surface will be tangent to the hyperbola. The element XY is parallel to the plane MN, and therefore its front view x' y' is an asymptote of the hyperbola. There are two elements of the surface having *xy* for a top view, but different front views both lettered $x'\ y'$. If both of these lines are rotated about AB as an axis they will each generate the same sur-

face. It is evident from this that **there are two rectilinear elements through every point of the surface.**

141. To represent an element of the surface. Draw the top view of any rectilinear element as ce tangent to the top view of the circle of the gorge. The front view of C is on the lower base at c' and that of E on the upper base at e'. Then $c'\ e'$ is the front view of the rectilinear element. The intersection o' of $c'\ e'$ with the front view of the circle of the gorge should be directly below the point of tangency o. To represent a circle of the surface, draw its top view p_1pp_2 and from this determine its front view $p_1'\ p_2'$. There is another circle of the surface, the distance above the circle of the gorge that $p_1'\ p_2'$ is below it, which has the same top view p_1pp_2. The front view of the circle could have been drawn first and then the top view found.

To represent a point of the surface. Select either the top or front view of the point and through it draw the corresponding view of either the rectilinear or circular element. Find the other view of the element, and then the corresponding view of the point on it.

142. To represent a plane tangent to the hyperboloid of revolution of one sheet at a point of the surface.

Let the surface be given as in Fig. 81, and let P be the point of tangency represented as in the preceding article.

Analysis. Through each point on the surface of a hyperboloid of revolution of one sheet there are two rectilinear elements. These two straight lines represent the tangent plane to the surface at their point of intersection. Through each point P of the surface there is also a circle and the straight line tangent to the circle at P lies in the tangent plane to the surface at P.

Construction. Through the top view of P, draw pe and pr tangent to the top view of the circle of the gorge. These are the top views of the rectilinear elements through P. The front views of these elements are $p'e'$ and $p'r'$. PE and PR represent the tangent plane to the surface at the point P. SS, the tangent to the circular element through P, also lies in the tangent plane to the surface at P.

Since the line CG cuts the lower base circle and ER the upper base circle of the surface, it is evident that the plane CEGR cuts the surface of the hyperboloid. It is tangent to the surface at the point P only. Each line on the surface through P, either straight or curved, will have its tangent at this point lying in the plane PER. The plane is therefore tangent to the surface at this point. It is a characteristic of a tangent plane to a warped surface that **it contains the rectilinear element of the surface through the point of tangency, but the plane is tangent to the surface at but one point of the element.**

HELICOIDS

143. A helicoid is a surface formed by a straight line moving about an axis in such a manner that each point of the moving line traces a helix; all the helices must have the same axis and the same pitch. The generating line may or may not intersect the axis of the surface but can not be parallel to it.

If the generatrix CE, Fig. 82, forms an acute angle with the axis AB, an **oblique helicoid** is formed, as in the triangular screw thread. If CE is perpendicular to the axis, a **right helicoid** is formed, as in the square screw thread. The helicoid is also used in the construction of screw conveyors.

From the nature of the motion it is evident that in general two successive positions of the generatrix are not in the same plane, therefore the surface is warped. In the particular case when the generatrix is tangent to a helix traced by one of its points, the surface generated is a **developable helicoid** or **helical convolute** which is a developable surface.

144. To represent the surface.

Let AB, Fig. 82, be the axis, CE the initial position of the generating line, and the distance $a'\ b'$ the pitch of all the helices traced by points of the generatrix CE. $a'q'o'\ y'\ b'$ is the front view of the helix traced by the point of the generatrix O which is nearest the axis. The views of

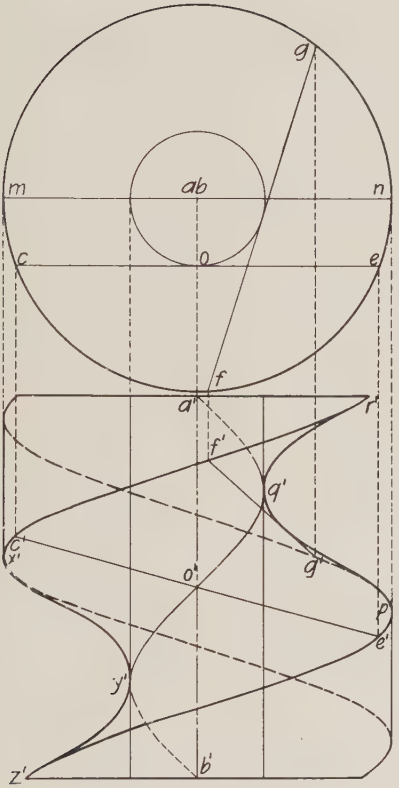


FIG. 82.—*The helicoid.*

the helix are drawn by the method of Art. 116. In a similar manner the helices $r'\ f'\ c'$ . . and $g'\ e'\ z'$ . . traced respectively by the points C and E are represented. From the nature of the motion of the generatrix CE, it is evident that its top view in every position is tangent to the circle with radius ao. Therefore to represent a rectilinear element of the surface, draw its top view fg tangent to the circle with radius ao. The front view of F is at f' on the front view of the helix traced by the point C. Likewise, the front view of G is at g' on the helix traced by the point E. Then $f'\ g'$ is the front view of an element of the surface. The front views of other elements are drawn in a similar manner and then the curve $p'\ q'\ r'$ is drawn tangent to the front views of these elements. In a similar manner the curve $x'\ y'\ z'$ is determined. These curves can also be determined by drawing them tangent to helices traced by different points of the generating line CE. The curves show the contour of the helicoid in the front view.

The portion of the helicoid usually represented is generated by a line of definite length as CE, and the top view of such a surface is the two concentric circles shown in

Fig. 82. Sometimes the generating line is extended indefinitely in one direction from the point O, and the intersection of the surface with a horizontal plane, as the one through B, taken as the base. The base in this case is a spiral. Points of the spiral are found first in the front view, where the elements of the surface pierce the plane of the base, and are then projected to the top view where the spiral shows in its true size.

The helicoid can be varied in form by changing the distance the generatrix CE is from the axis AB, the angle CE makes with the axis, or the pitch of the helix a' o' b'. In the surface of the ordinary screw thread, the generatrix cuts the axis. Both surfaces of the screw thread may or may not be generated by one line or this line prolonged beyond the axis.

145. To represent a plane tangent to a helicoid at a point of the surface.

Let the surface be represented as in Fig. 82, and let G be the given point.

Analysis. Since the surface is ruled, the rectilinear element through the point of contact lies in the tangent plane. A straight line tangent at the point of contact to any other line on the surface will also lie on the tangent plane. The simplest curve that can be drawn on the surface through the point of contact is a helix. The tangent plane, therefore, can be represented by the rectilinear element of the surface and the straight line tangent to the helix through the point of contact.

Construction. From g, draw a tangent to the circle with radius ao and extend the tangent to f. fg is the top view of a straight line element of the helicoid through the point G. The front views of F and G are on the front views of the respective helices traced by these points. $f'g'$ is the front view of the straight line element of the surface through G. By means of Art. 117 draw at G a straight line tangent to the helix traced by the point G. This tangent with the element FG represent the tangent plane to the surface at the point G. If the point at which the tan-

gent plane is to be drawn is not on one of the helices already drawn, it will be necessary to draw the views of a helix on the surface of the helicoid which passes through the point and then draw a straight line tangent to the helix at that point.

Since the helicoid is a warped surface, the plane which is tangent to it at the point G will not be tangent at any other point of the element FG.

HYPERBOLIC PARABOLOIDS

146. A hyperbolic paraboloid is the path of a straight line which, as it moves, touches two straight lines which are not in one plane and remains parallel to another given plane.

The moving line is called the **generatrix**, the two fixed straight lines the **directrices**, and the given plane the **plane directer**. Any position of the generatrix is called an **element of the surface.**

This surface has a second rectilinear generation in which any two rectilinear elements of the first generation may be taken as directrices and a plane which is parallel to the first directrices as a plane directer. It follows that **through any point of a hyperbolic paraboloid, two rectilinear elements can always be drawn.**

Any intersection of the surface by a plane, which is not a straight line intersection, is a hyperbola or a parabola, hence the name of the surface.

It is evident from the nature of the motion that two consecutive positions of the generating line are not in a plane, therefore the surface is warped.

The pilot or "cow catcher" of an American locomotive is usually of the form of the hyperbolic paraboloid.

147. To represent the surface. In Fig. 83, AB and CD are the rectilinear directrices and V is taken for the plane directer. The plane directer must either be indicated or represented. Rectilinear elements are drawn parallel to the plane directer and cutting the directrices. In this figure the top views of the elements are first drawn and then

their front views determined. A plane parallel with the plane directer will always cut the directrices in two points and a straight line joining these points will be an element of the surface. *mn* and *m′ n′* are the top and front views respectively of an element of the surface.

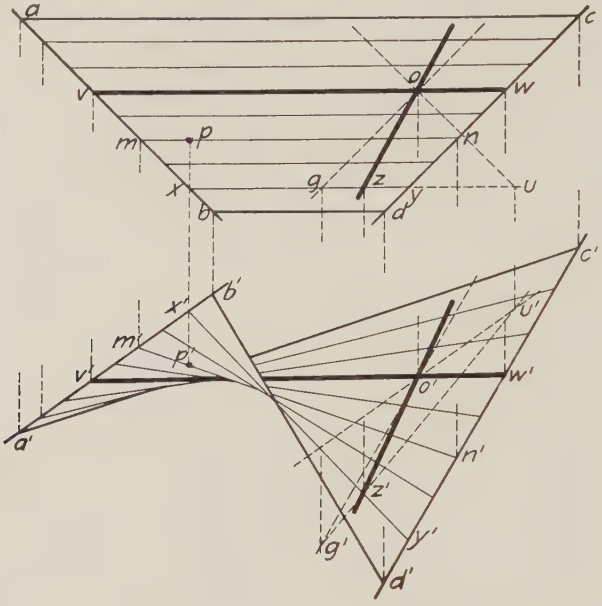


FIG. 83.—*Hyperbolic paraboloid and tangent plane.*

To represent a point of the surface, a rectilinear element, as MN, Fig. 83, is first drawn and then a point P of this element is taken.

148. To represent a plane which is tangent to the surface at a point of the surface.

Let O, Fig. 83, be the given point.

Analysis. Through the given point pass planes parallel respectively to the elements of each generation. These planes will intersect the surface in two rectilinear elements which determine the tangent plane at the given point.

Construction. The plane through O parallel to the elements of the first generation will cut the element VW from the surface. Through O draw OG parallel to CD, and OU

parallel to AB. These lines will determine a plane parallel to the elements of the second generation. This plane will intersect the element XY in Z. The line joining Z with O will be an element of the second generation. OW and OZ will therefore determine the required tangent plane to the surface at the point O. This plane is not tangent to the surface at any other point.

CONOIDS

149. A conoid is the path of a straight line which, as it moves, touches two other lines, one straight and the other curved, and remains parallel to a given plane.

When the generatrix of a right helicoid touches the axis, the helicoid is also a conoid. In this form the conoid has many practical applications.

To represent the surface. In Fig. 84, MN is the straight line and ABCD the curved line directrix. V is the plane directer. In this case, the top view of MN and the circle ABCD represent the top view of the surface. The front view of the surface is the triangle $m'\ a'\ c'$.

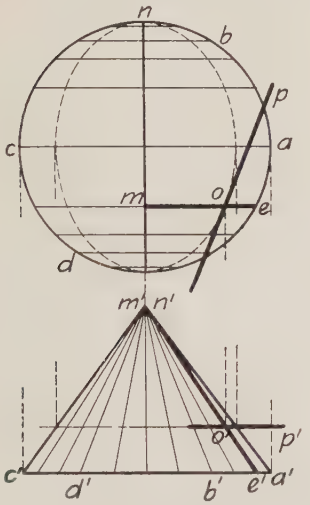


FIG. 84.—*Conoid and tangent plane.*

An element of the surface is represented by selecting some point of the curved directrix as E and drawing a straight line through this point parallel to the plane directer until it cuts the directrix MN at M. Then em is the top and $e'\ m'$ the front view of an element of the surface.

To represent a point of the surface, an element is first represented and then the point placed on the element. o and o' are the views of a point represented in this manner.

150. To represent a plane which is tangent to the conoid at a given point on the surface.

Let O, Fig. 84, be the given point.

Analysis. The tangent plane to the surface at the point O is determined by the straight line element through the point and a straight line tangent at O to some curve on the surface. There are many curves on the surface through O. One of the easiest curves to draw is cut from the surface by a plane parallel to the plane of the base of the surface.

Construction. The top view *me* of the straight line element is drawn first through O parallel to V, and then the front view *m'e'* is determined. ME is one line of the tangent plane. The horizontal plane through O cuts an ellipse from the surface. The top view of this ellipse is drawn through the top views of the points in which the straight line elements of the surface pierce the horizontal plane through O. OP is drawn tangent to this ellipse. EM and OP represent the tangent plane to the surface at the point O. Since the conoid is a warped surface, the plane MEOP is tangent to the surface at the point O only.

CYLINDROIDS

151. A cylindroid is the path of a straight line which, as it moves, touches two curved lines and remains parallel to a given plane. The surface is used in architecture for joining the arched ceilings in two parallel corridors of different levels.

To represent the surface. In Fig. 85, ABC and DEG are the curved directrices and H is the plane directer. The surface is represented in the front view by the curved directrices and a series of horizontal lines. The top view of the surface consists of the top views of these straight line elements and the curved directrices. If enough straight line elements are drawn in the top view, the contour of the surface will show as a curved line touching all the top views of these elements.

An element of the surface is represented by selecting some point B of one directrix and drawing a straight line

parallel to the plane directer H until it cuts the other directrix at E. Then b' e' is the front and be the top view of a rectilinear element of the surface.

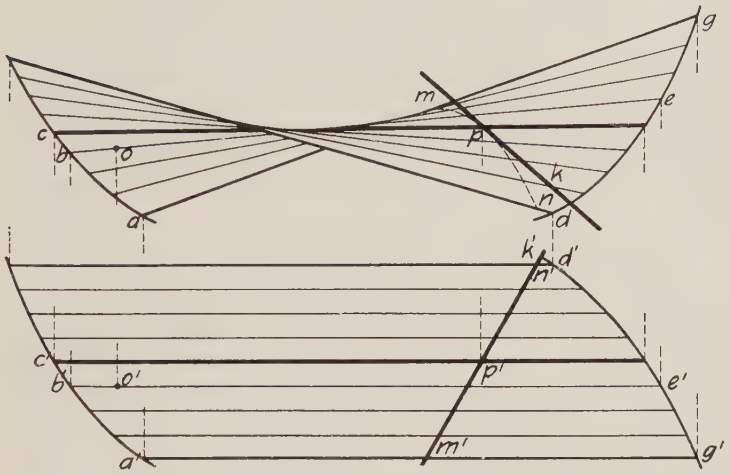


FIG. 85.—*Cylindroid and tangent plane.*

A point of the surface is represented by first representing an element of the surface as BE and then taking a point O of this element . o and o' are the views of a point on the surface.

152. To represent a plane which is tangent to the cylindroid at a point of the surface.

Let P, Fig. 85, be the given point.

Analysis. The tangent plane to the surface at the point P is determined by the straight line element through the point and a straight line tangent at P to some curve on the surface. There are many curves on the surface through P. The easiest curves to construct are those cut from the surface by planes perpendicular to either H or V.

Construction. Draw $c'p'$ through p' parallel to G.L. The top view cp is determined by the points in which the element CP cuts the curved directrices ABC and DEG. CP is one line of the plane tangent to the cylindroid at P.

Draw $m'n'$ in any direction with the exception of parallel to G.L. This is the front view of a curve on the surface of the cylindroid; the plane of the curve is perpendicular to V. The top view $m\ n$ of this curve is drawn through the top views of the points in which the straight line elements pierce the plane of the curve. PK is the tangent to the curve MN at the point P. Then CP and PK represent the tangent plane to the cylindroid at the point P. Since the cylindroid is a warped surface, the plane CPK is tangent to the surface at the point P only.

DOUBLE CURVED SURFACES

153. A double curved surface is a surface which can be generated only by the motion of a curved line. Such a surface has no straight line elements. The surface may be double convex as a sphere or ellipsoid, concave outward as the curved surface of a pulley, or concavo-convex as the annular torus or anchor ring.

The more common form of a double curved surface is one of revolution; that is a surface which has no straight line elements and can be formed by the rotation of some curved line about an axis. The general form of the double curved surface, however, is not one of revolution. For example, the general form of the ellipsoid is a surface in which the three principal axes through the center are all of different length, while in the ellipsoid of revolution two of these axes are of equal length.

Since the general form of the surface is not common in practical work, **only double curved surfaces of revolution will be considered in this text.**

154. Some of the simple double curved surfaces of revolution are the following:

A sphere is the path of a circle revolving about its diameter as an axis.

An annular torus or anchor ring is the path of a circle revolving about a straight line which lies in the plane of the circle but does not cut the circumference.

An ellipsoid of revolution is the path of an ellipse revolving about either axis. When the ellipse is revolved about its major axis, a **prolate ellipsoid** is generated; when about its minor axis, an **oblate ellipsoid**. These surfaces are sometimes called **spheroids.**

A paraboloid of revolution is the path of a parabola revolving about its axis. This surface is extensively used for light reflectors.

A hyperboloid of revolution of two sheets is the path of a hyperbola revolving about the axis which passes through the foci.

155. To represent a double curved surface of revolution. For convenience of representation, surfaces of revolution are usually taken with the axis of rotation perpendicular to one of the principal planes of projection, although the surface may be placed in any position. The view of the surface taken by looking in the direction of the axis of rotation is one or more circles. If the surface is limited the largest circle is shown, but if the surface is indefinite in extent its intersection with some plane at right angles to the axis is shown. The view of the surface taken by looking at right angles to the axis is the meridian line, the plane of which is parallel to the picture plane. A meridian line of a surface of revolution lies in a plane which contains the axis of the surface. For example, a prolate ellipsoid with its axis of rotation perpendicular to H, will have for its top view a circle with diameter equal to the minor axis of the generating ellipse and for its front view an ellipse identical to the generating ellipse, and having its major axis perpendicular to G. L.

Since there are no straight lines on a double curved surface of revolution, a circle of the surface with its plane perpendicular to the axis may be considered the element. A view of the element looking in the direction of the axis is a circle, and the view looking at right angles to the axis is a straight line equal in length to the diameter of the circle, and having its extremities in the contour of the surface.

If one view of a point on a surface of revolution is given, the other view can be found as follows: Through the given view of the point draw the corresponding view of the circular element of the surface. Then find the other view of the element and place the required view of the point on it. It must not be overlooked that if the top view shows the true size of a circular element, the front will show it as a straight line parallel to G. L. and vice versa.

156. Tangent planes to double curved surfaces of revolution. The tangent plane to a surface at a point on the surface is determined by two straight lines tangent at this point to two lines of the surface. The simplest curves passing through the point on a surface of revolution are usually the meridian line and the circle which is cut from the surface by a plane perpendicular to the axis of the surface. The tangent plane, therefore, is determined by two straight lines, one tangent to the circular section at the given point and the other tangent to the meridian section at that point.

157. To represent a plane which is tangent to an ellipsoid of revolution at a point of the surface.

Let the surface be given as in Fig. 86, and let P be the given point.

Analysis. A tangent to the meridian curve at the given point and a tangent to the circle of the surface at this point will represent the required tangent plane.

Construction. $x\ p\ r$ is the top and $x'p'r'$ the front view of the circle of the surface through the given point P. If the top view of P had been given the front view would have been found by drawing the circle $x\ p\ r$, finding the front view $x'r'$ of this circle, and then locating p' on $x'r'$. If the front view of P had been given the top view would have been found by drawing the front view $x'p'r'$ of the circle, finding the top view $x\ r$ of this circle, and then locating p on the circle $x\ p\ r$. $p\ m$ is the top and $p'm'$ the front view of the tangent to the circle at the point P. This is one line of the required tangent plane. AYZB is the me-

ridian section through P of the ellipsoid. The front view of this section is an ellipse with $y'z'$ for a major and $a'b'$ for a minor axis. With the axes of the ellipse given the foci can be determined by the method of Art. 109 and then

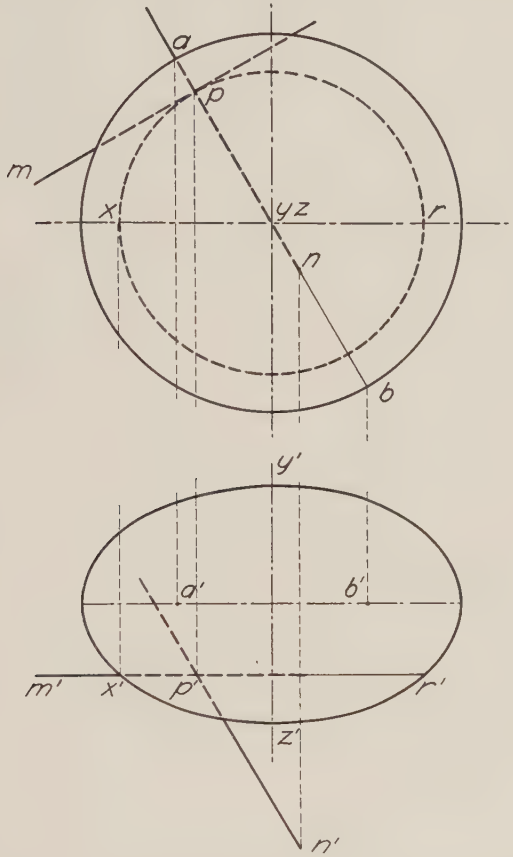


FIG. 86.—*Ellipsoid of revolution and tangent plane.*

the tangent $n'p'$ constructed by the method of Art. 110. It is not necessary to draw the front view of the ellipse. Then NP is tangent to the meridian section of the surface at the point P. The required tangent plane is represented by the lines PM and PN.

The tangent PN to the meridian section could have been constructed by means of an auxiliary view showing the true size of the ellipse cut from the surface by the meridian plane. This section would be the same size as the front view of the surface. Another method of getting the tangent PN is to revolve the meridian section AYZB about YZ as an axis until the plane of the section is parallel with V. In this position the meridian section appears as the front view of the surface, the point P coinciding with X. The tangent to the ellipse can be drawn at X and then the meridian section with the tangent revolved to the original position of the meridian plane.

There are any number of planes tangent to a double curved surface of revolution from a point outside the surface. One or more of these planes can be determined by means of a plane through the given point and perpendicular to the axis of rotation, providing this plane cuts the surface. The tangent from the given point to the circle cut from the surface by this plane will locate the point of tangency of the tangent plane. With the point of tangency located, the problem is the same as that solved in the above paragraph. Since there are two tangents to the circle from the given point, two tangent planes can be determined by this method. Other tangent planes can be drawn by means of the meridian plane which contains the given point and the axis of the surface. Tangents from the given point to the meridian section will determine the points of tangency of the planes and the planes can then be constructed as in the preceding paragraph.

158. Problems.

Solve each of the following problems in a 7½" x 11" rectangle with the base line through the center and parallel to the shorter sides of the rectangle.

1. O (0, 2, —3) is the center of a sphere (3" diameter). (a) Represent a plane which is tangent to the surface at the point P (—¾, 1, ?) (b) Represent a plane which contains the point M (1¼, ¼, —¼) and is tangent to the sphere.

2. A (0, 2½, —¾) B (0, 2½, —4¾) is the major axis of a prolate ellipsoid of revolution whose minor axes are 2½". (a) Represent a plane which is tangent to the ellipsoid at the point P (—½, 1½, —1¾). (b) Represent a plane which contains the point M (1½, 1, —1) and is tangent to the ellipsoid.

3. A (—2, 2½, —1¼) B (2, 2½, —1¼) is the directrix of a parabola whose focus is F (0, 2½, —2). Construct the top and front views of the paraboloid formed by rotating this parabola about its axis. (a) Represent a plane which is tangent to the paraboloid at the point P (—¾, ?, —3). (b) Represent a plane which contains the point M (1½, 2, —4) and is tangent to the paraboloid.

4. A (—1⅜, 2½, —3½) is the center of a circle (1¼" diameter) which lies in a vertical plane. Construct the top and front views of the torus formed by rotating this circle about X (0, 2½, —2) Y (0, 2½, —5) as an axis. (a) Represent a plane which is tangent to the torus at the point P (—1, 1⅛, ?). (b) Represent a plane which contains the point M (2¼, ⅝, —1¾) and is tangent to the torus.

159. To represent a plane which contains a given straight line and is tangent to a sphere.

Let AB, Fig. 87, be the given straight line and let C be the center of the given sphere.

Analysis. Take an auxiliary view in the direction of the given line. In this view the line appears as a point and the

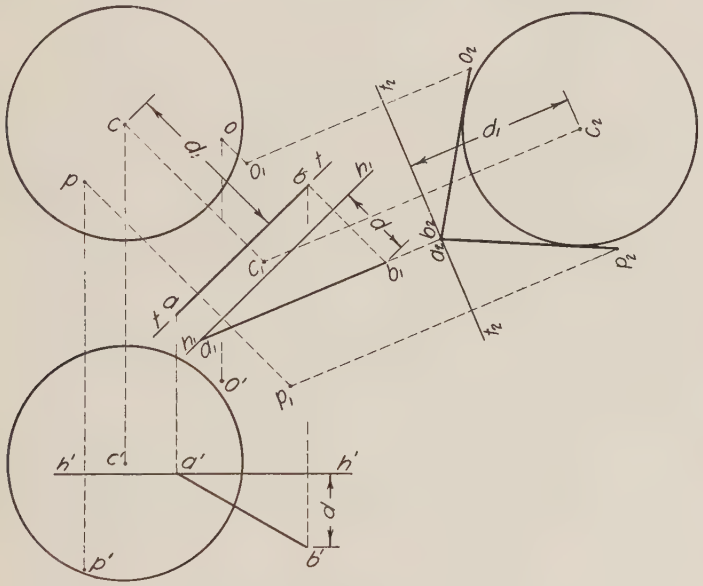


FIG. 87.—*Plane containing line AB and tangent to sphere.*

required tangent plane appears as a straight line through this point and tangent to the circle which represents the sphere. If the given line does not touch the sphere, there are two solutions to the problem.

Construction. In the first auxiliary view, c_1 is the center of the sphere and a_1b_1 is the true length of the given line AB. In the second auxiliary view, c_2 is the center of the sphere and the point a_2b_2 represents the line AB. With c_2 as center and the given radius of the sphere as a radius, draw a circle. This circle represents the sphere in the second auxiliary view. a_2o_2, tangent to the circle, represents the required tangent plane in the second auxiliary view. $a_1b_1o_1$ is the first auxiliary, $a\ b\ o$ the top, $a'b'o'$ the front view, respectively, of the required tangent plane. ABP is also a plane containing the line AB and tangent to the given sphere.

If the line AB had been tangent to the sphere there would have been but one solution to the problem. If AB had cut the surface of the sphere, there would not have been a solution to the problem.

160. Problems.

Solve each of the following problems in a 7½″ x 11″ rectangle with the base line through the center and parallel to the shorter sides of the rectangle.

1. Represent a plane which contains the line F (—⅞, ½, —1⅜) F (⅞, ½, —2¼) and is tangent to a sphere of 1″ radius with center at O (—1½, 2, —1½).

2. Represent a plane which contains the line S (—2, 3¾, —4¼) S (—¼, 2, —4¼) and is tangent to a sphere of 1″ radius with center at O (—1½, 2½, —2½).

3. Represent a plane which contains the line M (—2½, 2, —1) N (¼, 2, —1) and is tangent to a sphere of 1″ radius with center at O (—1, 1½, —2½).

4. Represent a plane which contains the line M (—1, 1¼, —2¾) N (—1, 2⅞, —4⅜) and is tangent to a sphere of 1″ radius with center at O (—1½, 2½, —2).

5. Represent a plane which contains the line E (3, 1½, —3½) G (—¾, 3½, —1) and is tangent to a sphere (2″ diameter) with center at O (—1¾, 2, —4¼).

GENERAL PROBLEMS INVOLVING TANGENT PLANES TO CURVED SURFACES

161. Problems.

Solve each of the following problems in a 7½″ x 11″ rectangle with the base line through the center and parallel to the shorter sides of the rectangle.

1. Represent a plane which is tangent to a sphere (1½″ diameter) with center at O (—¼, 3⅜, —1¼) and is parallel to the plane A (—1½, 3⅜, —2⅝) B (0, 1⅞, —2⅝) C (—1¾, 1⅞, —1¼).

2. Inscribe a sphere in the tetrahedron V (—½, 3½, —2½) C (—2½, 4¼, —5) E (—2½, 1½, —5) G (¼, 1½, —5).

3. A (—1½, 1, —2¾) B (—¼, 1⅝, —1⅜) C (½, ½, —1⅞) is a plane which is tangent to a right circular cylinder whose axis is E (—¼, 1⅜, —2⅝) G (1, 2, —1¼). Draw the top and front views of the element of contact.

4. X (—2¼, 1, —5) Y (1, 3, —1) and M (1, 3½, —1½) N (2½, 1, —5) are the axes of two oblique cylinders whose bases are horizontal circles (2″ in diameter). To the two given cylinders construct tangent planes which are parallel to each other.

5. X (—2¼, 1, —5) Y (1, 3, —1) is the axis of an oblique cylinder whose base is a circle (2″ diameter) in H. M (1, 3½, —1½) N (2½, 1, —5) is the axis of an oblique cone whose base is a circle (2″ diameter) in H with center at N. To the given cone and cylinder construct tangent planes which are parallel to each other.

6. A (—2¼, 1, —5) B (—½, 3, —2¼) and C (½, 2½, —2¼) D (2½, 1, —5) are the axes of two cones whose bases are circles (2″ and 2¼″ diameters respectively) in H, with centers at A and D. To the two given cones construct tangent planes which are parallel to each other.

7. E (3, 1½, —3½) G (—¾, 3½, —1) is the axis of a right circular cylinder (1½″ diameter). O (—1¾, 2, —4¼) is the center of a sphere (2″ diameter). Construct a common tangent plane to the sphere and the cylinder. (The top and front views of the cylinder are not required.)

8. Give the analysis of the problem to pass a plane tangent to three given spheres.

CHAPTER VI.

PLANE SECTIONS AND DEVELOPMENTS OF CURVED SURFACES

162. Plane section. The line of intersection of an oblique plane with a curved surface which is ruled can be drawn by finding the points in which the rectilinear elements of the surface pierce the given oblique plane. The line joining in proper order these piercing points is the required line of intersection of the surface with the plane. These piercing points can be found by either of the methods given for finding where a straight line pierces a plane, but the auxiliary view method is the better. The auxiliary view should be taken from such a direction that the cutting plane will appear as a straight line. The rectilinear elements should be taken near enough together to give the line of intersection as accurately as desired.

The intersection of an oblique plane with a double curved surface is found by using a system of auxiliary planes. This is the method of Art. 57, where the given line is a plane curve rather than a straight line. Each of the auxiliary planes will cut a straight line from the oblique plane and a curved line from the double curved surface and these lines will intersect in points of the required line of intersection. The auxiliary planes should be so taken that they will cut simple curves from the double curved surface and these curves should be in simple positions with reference to the planes of projection.

Development. By the development of a curved surface is meant a plane area of such form and size that it can be rolled or folded to form again the original surface. The development is obtained by rolling the surface upon a tangent plane until each part of the surface comes in contact with the plane. The part of the plane thus covered is the development of the surface.

163. To find the intersection of a right circular cylinder with a given oblique plane and to develop the surface of the cylinder.

Let the cylinder be given as in Fig. 88, and let SS and FF represent the oblique plane.

Analysis. Find the points in which the rectilinear elements of the cylinder intersect the given oblique plane. A curved line joining these points in the proper order is the required line of intersection.

Construction. A view taken by looking in the direction of the horizontal line SS, shows the cylinder as a rectangle and the oblique plane as the line $s_1\ f_1$. The distance

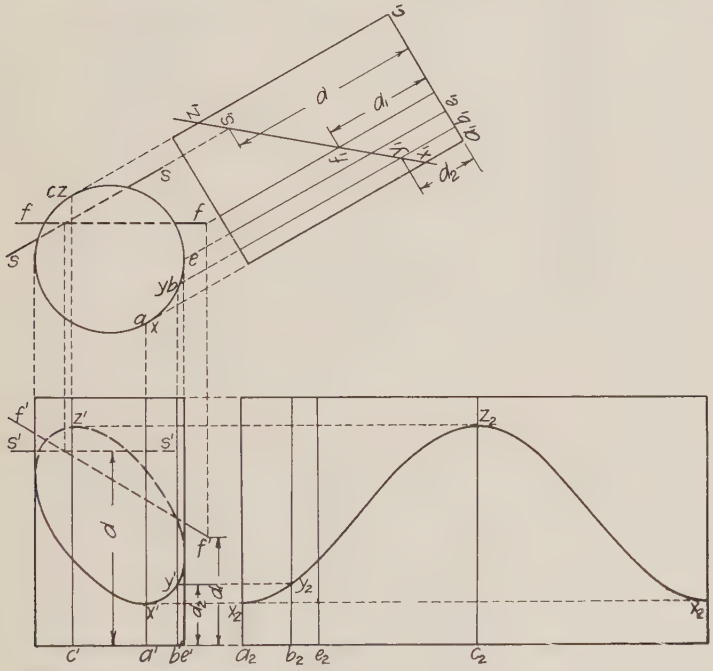


FIG. 88.—*Plane section and development of a right circular cylinder.*

at which the line $a_1\ c_1$, representing the base of the cylinder, is placed from the top view is arbitrary. The distances d and d_1 are taken from the front view and used to locate the line $s_1\ f_1$ which represents the oblique plane in

the auxiliary view. The front and auxiliary views of elements of the cylinder are drawn from points in the base as A, B, C, etc., as many elements being drawn as are necessary to determine the curve of intersection to the desired accuracy. The points x_1, y_1, z_1, etc., where the auxiliary views of these elements intersect the line s_1 f_1 are the auxiliary views of the points in which the elements pierce the cutting plane SS, FF. By taking the distances of the points x_1, y_1, z_1, etc., from the line a_1 c_1 and setting them off on the proper elements from the base a' c', the front view x', y', z', etc., of the points are located. Joining these points in the proper order gives the ellipse which is the front view of the intersection of the cylinder with the plane. The top view of the intersection is the circle a, b, c.

To develop the surface of the cylinder showing the curve of intersection with the oblique plane. If the cylinder be rolled on a tangent plane until each element has come into the plane, the base will develop into the straight line a_2c_2. The distance a_2 b_2=arc ab, b_2 e_2=arc be, etc. In the development, the elements will be perpendicular to a_2 a_2, since they are perpendicular to the plane of the base. To get the line of intersection on the development, lay off a_2 x_2 $=a'$ x', $b_2y_2=b'y'$, etc. A smooth curve passing through the points x_2, y_2, z_2, x_2, is the development of the line of intersection.

164. To draw a straight line tangent to the curve of intersection of a plane with a curved surface at any point on the curve. The curve is evidently a plane curve and therefore a tangent to it at any point must lie in the plane which cuts the curve from the surface. Since the curve lies on the surface, a line which is tangent to it at any point must lie in a tangent plane to the surface at that point. The required line is, therefore, the line of intersection of the cutting plane with the plane tangent to the surface at the point to which the tangent line is to be drawn.

This method is general and applies to the plane section of all surfaces. The construction for the problems is not

as a rule, difficult, since usually the tangent plane to the surface can be drawn easily.

In this chapter of the text, the lines tangent to the plane sections of the different surfaces will not be shown in the figures, but in every case they can, if desired, be drawn by the above method.

165. To find the intersection of an oblique cylinder with a given oblique plane.

Let the cylinder be given as in Fig. 89, and let SS and FF represent the oblique plane. This plane is at right angles to the axis of the cylinder but the method of finding the curve of intersection of the cylinder with the plane is the same, regardless of the position of the plane with respect to the axis of the cylinder. The plane was taken at right angles to the axis in order to use this figure in explaining the development of the cylinder as given in the next article.

Analysis. Take a view of the cylinder and plane by looking in the direction of the horizontal line SS. This view shows the cutting plane as a line. The points in which this line crosses the auxilary view of the elements of the cylinder are the auxiliary views of points on the required curve of intersection. From the auxiliary view, the top and front views of the intersection can be derived.

Construction. In the auxiliary view, the base of the cylinder appears as the line $a_1\ b_1$ at right angles to ss, and the center of the base as the point c_1. The auxiliary view $c_1\ o_1$ of the axis is determined and then the extreme elements of the cylinder are drawn parallel to $c_1\ o_1$ through the points a_1 and b_1. The upper end of the cylinder is shown as broken. $s_1\ f_1$ is the auxiliary view of the oblique plane. The points where the auxiliary view of the elements cross the line $s_1\ f_1$ are the auxiliary views of the points where these elements pierce the cutting plane. The top views of these piercing points are found by first finding the top views of the elements and then locating the top views of the points on the top views of the corresponding elements. From the auxiliary and top views, the front views of the

piercing points are found. It is frequently more accurate to use the distance d_1 taken from the auxiliary view to locate the front view of a point as X than to project from the top view of X to the intersection with the front view of the element through X. A curved line joining these points in the proper order is the required line of intersection.

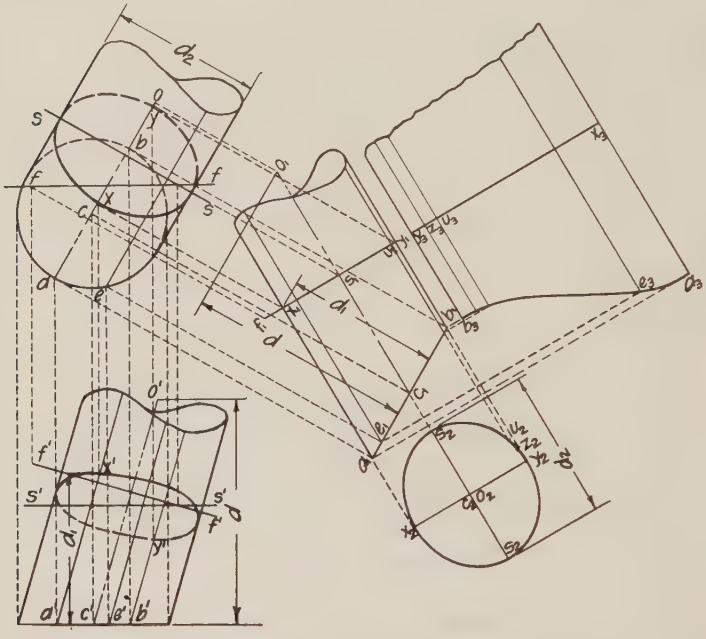


FIG. 89.—*Plane section and development of an oblique cylinder.*

166. To develop the surface of an oblique cylinder. Let the cylinder be given as in Fig. 89.

Analysis. If the cylinder be rolled on a tangent plane, the section which is at right angles to the axis will become a straight line on the development and the elements of the cylinder will appear in their true lengths and lie at right angles to this line. Therefore, take an auxiliary view showing the true lengths of the elements of the cylinder and then a second auxiliary view showing the true size of a right section of the cylinder. From these two views the development can be constructed.

Construction. The first auxiliary view shows the true length, $c_1\ o_1$, of the axis and also the true lengths of all the elements of the cylinder. The second auxiliary view shows the cylinder as an ellipse. The minor axis $x_2\ y_2$ is equal to the width $x_1\ y_1$ of the first auxiliary view while the major axis is equal to the width d_2 of the top view. The ellipse can be constructed by the trammel method or by getting the views of points where the elements of the cylinder cut the curve of the base. From the true size of the right section the distances are laid off as follows along a straight line $y_3\ x_3$: arc $y_2\ z_2 = y_3\ z_3$, arc $z_2\ u_2 = z_3\ u_3$, etc. The true lengths of the elements are taken from the first auxiliary view and laid off on the development at right angles to $y_3\ x_3$ at the proper points along that line. In Fig. 89, the line $y_3\ x_3$ in the development was taken in the continuation of $x_1\ y_1$ in order that the true lengths of the elements could be projected from the first auxiliary view to the development. In the development, the line $y_3\ x_3$ represents the right section and $b_3\ e_3\ a_3$ the curve of the base of the cylinder. Only half the cylinder is shown in the development.

It is not necessary to construct the top and front views of the right section in order to develop the surface of the cylinder.

167. Problems.

Solve each of the following problems in a 11″ x 16″ rectangle with base line through the center and parallel with the longer sides of the rectangle.

1. The base of a right circular cylinder which has its axis parallel to H and 60° with V is a 2½″ circle with center at A (—5¼, ¾, —3). A horizontal line of a plane which makes 45° with H is perpendicular to the elements and cuts the axis 2½″ from A. Construct the top and front views of the section cut from the cylinder by this plane. Develop the surface of the cylinder, showing the curve of intersection with the plane on the development.

2. The base of an oblique cylinder is a 2½″ circle in H with center at C (—5½, 1, —4¾). The top view of the axis makes 60° with the ground line and the axis passes through O (—4, ?, —¾). Develop the surface of the cylinder.

3. A (—4½, 2¾, —½) B (—4½, 2¾, —5) is the axis of a right cylinder whose base is an ellipse in H (major axis 3½″, minor axis 2″). (a) Find the top and front views of the line of intersection of this cylinder with the plane M (—6, 2½, —3½) N (—4½, 4½, —1) O (—2¼, 1, —2¾). (b) Represent a plane which will cut a circle from this cylinder with diameter equal to the major axis of the base.

168. To find the intersection of a cone with a given oblique plane.

Let the cone be given as in Fig. 90, and let SS and FF represent the oblique plane. Fig. 90 shows an oblique cone but the method is the same for a right circular cone.

Analysis. Take a view of the cone and plane by looking in the direction of the horizontal line SS. This view shows the cutting plane as a line. The points in which this line

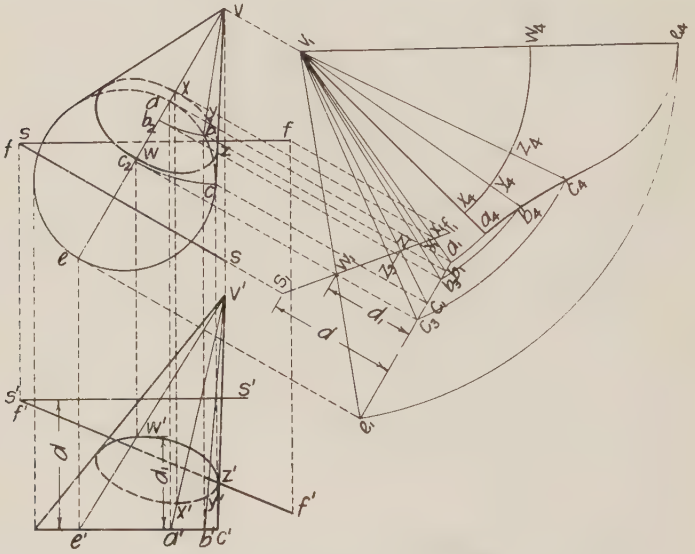


FIG. 90.—*Plane section and development of an oblique cone.*

crosses the auxiliary view of the elements of the cone are the auxiliary views of the points on the required curve of intersection. From the auxiliary view, the top and front views of the intersection can be derived.

Construction. $v_1\ a_1\ e_1$ is the auxiliary view of the cone and $s_1\ f_1$ that of the cutting plane. It is best to divide each half of the circumference of the base of the cone into the same number of equal parts, beginning at A, the foot of the shortest element. Then each element in the auxiliary view represents two elements of the cone, one visible and one invisible. By drawing the auxiliary views of the ele-

ments, the auxiliary views x_1, y_1, z_1, etc., of the points of intersection are found. From the auxiliary views, the top and then the front views are derived. A curve joining in proper order the points thus found, is the required line of intersection. The top and front views of this section are ellipses.

169. To develop the surface of a cone.

Let the cone be given as in Fig. 90.

Analysis. A cone is developed by drawing a sufficient number of elements to divide the base into reasonably small parts. The part of the surface of the cone included between two elements and the arc of the base is considered a plane and laid out on the development as a triangle having given the three sides. To develop the cone, therefore, it is necessary to have the true size of the base of the cone and the true lengths of the elements used in making the development. If the cone is right circular, it is necessary to have the length of but one element.

Construction. In Fig. 90, the base of the cone is in a horizontal plane so the top view shows the true size of the circle of the base. To make the development, use the elements of the cone VA, VB, VC, etc., which were used in constructing the plane section. These elements do not all show true length in any of the given views although two of them, VA and VE, show true length in the auxiliary view. The true lengths of other elements are found by rotating them about a vertical axis through V until they are parallel to the auxiliary picture plane. In this construction b in the top view moves to b_2 and b_1 in the auxiliary view to b_3, $v_1\ b_3$ being the true length of the element. With similar construction, $v_1\ c_3$ is found to be the true length of VC. By the same method, the length of any element can be found. To construct the development, draw a straight line in any direction through v_1 and lay off $v_1\ a_4 = v_1\ a_1$, the true length of the shortest element of the cone. With v_1 as center and $v_1\ b_3$ as radius strike an indefinite arc. With a_4 as center and radius equal to the rectified arc ab, strike an arc cutting the first arc at b_4. Then $v_1\ b_4$ is the element VB laid off on the development. By using v_1 as

center and v_1 c_3 as radius and b_4 as center and radius equal to the rectified arc bc, the point c_4 is located. By continuing the process the development as shown in Fig. 90 is found. The development for only half the cone is shown, the other half is similar. If the given cone had been right circular, the curve a_4 . . . e_4 would have been the arc of a circle with center at v_1.

To lay the curve of intersection of the cone with the plane SS FF off on the development, the true lengths of the elements from the vertex to the cutting plane are found and laid off on the corresponding elements of the development. This is easily accomplished by drawing lines through the points x_1, y_1, z_1, etc., parallel to a_1 e_1, until they cut the true lengths of the corresponding elements. The construction is shown for the point Z, v_1 z_3 being the true length from the vertex to the cutting plane. The distances thus found laid off on the development give the curve X_4 . . . W_4.

170. Problems.

Solve each of the following problems in a 11″ x 16″ rectangle with base line through the center and parallel with the longer sides of the rectangle.

1. The base of an oblique cone is a 2½″ circle in H with center at C (—5½, 1, —4¾) and vertex at V (—4, 3⅝, —¾). The top view of a horizontal line of a plane which makes 45° with H is perpendicular to the top view of the axis. Find the top and front views of the line of intersection of the plane and cone and develop the surface of the cone, showing the curve of intersection on the development.

2. A right circular cone has its axis parallel to H and 60° with V. The radius of the base is 1½″ and the altitude is 3½″. Find the top and front views of the intersection of this cone with a plane which makes 45° with H and has its horizontal line at right angles to the axis of the cone. Also develop the surface of the cone showing the curve of intersection on the development.

3. A horizontal circle 3″ in diameter is the base of an oblique cone. The axis of the cone is 5″ long, makes 60° with H and is oblique to V. Find the top and front views of the intersection of this cone with a plane perpendicular to the axis and develop the surface of the cone, showing the curve of intersection on the development.

4. From a right circular cone with axis A (—4, 2¾, —1) B (—4, 2¾, —5) and base a circle (3½″ diameter) in a horizontal plane through B, cut an ellipse, parabola, or hyperbola, as the instructor directs, and show the true size of the section.

5. Is the shortest path around the cone given in problem 4, following the surface, starting at the point P (—5, ?, —4) and returning to the same point, a plane curve? Construct the top and front views of this path.

171. To find the intersection of a warped surface with a given oblique plane.

Analysis. Since a warped surface is ruled, it is only necessary to find where the rectilinear elements of the surface pierce the plane. A line joining in proper order these points is the required line of intersection. The most convenient method of finding these piercing points is by means of an auxiliary view which shows the cutting plane as a line.

The construction is left as an exercise for the student.

A warped surface cannot be developed. For an approximate development of such a surface, see Art. 174.

172. To find the intersection of any surface of revolution with a given oblique plane.

Let the surface be given as in Fig. 91 and let AB and AC represent the oblique plane.

Analysis. Cut the surface and plane by a system of auxiliary planes perpendicular to the axis of revolution of the surface. The auxiliary planes will cut circles from the surface which will intersect the lines cut from the given plane in points of the required curve of intersection.

Construction. Since the method is the same for all surfaces of revolution, let the construction be made for the torus as given in Fig. 91. The axis of the surface is perpendicular to H. The auxiliary cutting planes will therefore be parallel to H, and will be represented in the front view by lines parallel to G. L. $h'\ h'$ represents one of these planes. It cuts the line SS from the plane ABC and two circles from the torus, one with a radius $q'\ r'$ and the other with a radius $q'\ u'$. The top view shows these circles cutting ss at x, y, z, and w, the top views of four points on the required curve of intersection. The front views of these points are on the line $h'\ h'$. By taking other planes parallel to H, enough points can be found to determine the curve of the intersection to the desired accuracy.

When the surface is symmetrical, as is the case with the torus, it is convenient to begin the construction with the plane which contains the center of the front view of the surface. This plane contains the largest and smallest circles of the torus as shown in the top view. If the remain-

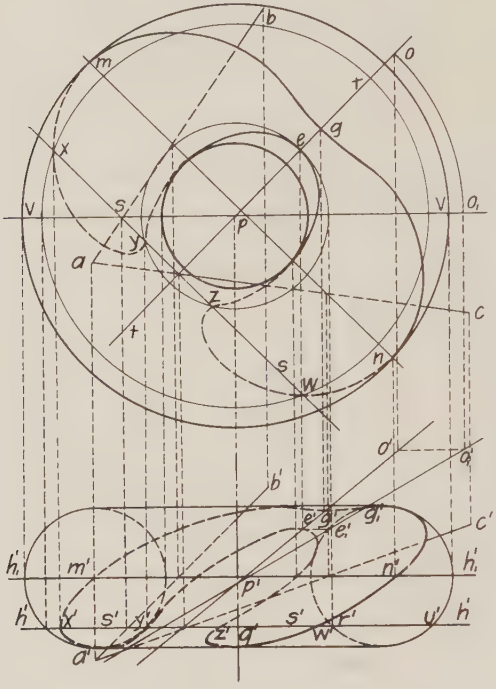


FIG. 91.—*Plane section of a torus.*

ing planes be passed in pairs at equal distances above and below the center plane, each pair will cut circles having the same top view. This will avoid the necessity of drawing so many circles in the top view. The highest and lowest planes which touch the surface will contain but one circle each.

A plane which contains the axis of the surface and is parallel to V, cuts a line from the plane ABC which intersects the small circles of the front view of the torus at the points where the curve of intersection touches these cir-

cles. A plane T which contains the axis of the surface and is perpendicular to a horizontal line SS of the plane ABC will cut a straight line OP from the plane ABC and two circles from the torus. The front view of OP is determined by the front views of the points in which two lines of the plane, as AC and BC, pierce the plane T. The circles cut from the torus by the plane T are the same size as those shown in the front view of the surface. The front views of these circles are ellipses. Instead of drawing the ellipses in the front view, the plane T can be revolved about the axis of the surface until T is parallel with V. In rotating, the point O moves to O_1 and P, being on the axis of rotation, remains fixed. In the rotated position, the circles cut by the plane T appear in the front view as the two circles representing the outline of the surface. $o_1'\ p'$ intersects one of these circles in the points e_1' and g_1'. By counter rotating the plane T, the required points E and G are located. Had the line $o_1'\ p'$ cut both circles, there would have been four points of intersection along the line $t\ t$ in the top view. Had $o_1'\ p'$ missed both circles, there would not have been any points of intersection along the line $t\ t$. For most surfaces of revolution, this section will locate the highest and lowest points on the curve of intersection of the oblique plane with the surface. Instead of rotating the plane T until it is parallel to V, the points E and G could have been located by taking an auxiliary view of the surface and plane ABC by looking in the direction of the line SS. This, however, would have involved more construction than the method by rotation.

173. Problems.

Solve each of the following problems in a 7½" x 11" rectangle with the base line through the center and parallel to the shorter sides of the rectangle.

1. Construct the top and front views of the line of intersection of the plane A (—2½, 1, —½) B (—½, 4½, —4) C (1½, 1, —3) and a sphere (3½" diameter) with center at O (0, 2½, —3).

2. A (0, 2½, —½) B (0, 2½, —5) is the major axis of a prolate ellipsoid of revolution whose minor axes are 3" long. Construct the top and front views of the line of intersection of the plane M (—2½, 3, —¼) N (0, ½ —¼) X (½, 4½, —4½) Y (3, 2, —4½) with the ellipsoid.

3. A (0, 2½, —⅞) B (0, 2½, —4½) is the axis of revolution of a paraboloid whose focus is at F (0, 2½, —1¼) and vertex at A. Construct the top and front views of the line of intersection of the plane X (—2½, 1¼, —1) Y (—1, 4¼, —4) Z (2½, 1½, —3½) with the paraboloid.

4. A circle (1¾" diameter) in V with center at O (—1⅝, 2½, —3½) is rotated about the axis X (0, 2½, —2) Y (0, 2½, —5) generating a torus. Construct the top and front views of the line of intersection of the plane A (2¾, 1, —2) B (—2, 0, —4) C (½, 5, —5) with the torus.

174. Approximate development of warped surfaces. Since two consecutive straight line elements of a warped surface do not lie in one plane, the surface cannot be developed. In practice, however, approximate developments of these surfaces are made in the following manner.

Let the surface be a transitional piece joining a circle at one level with an ellipse at a different level, Fig. 92. The openings joined by the transitional piece need not be in par-

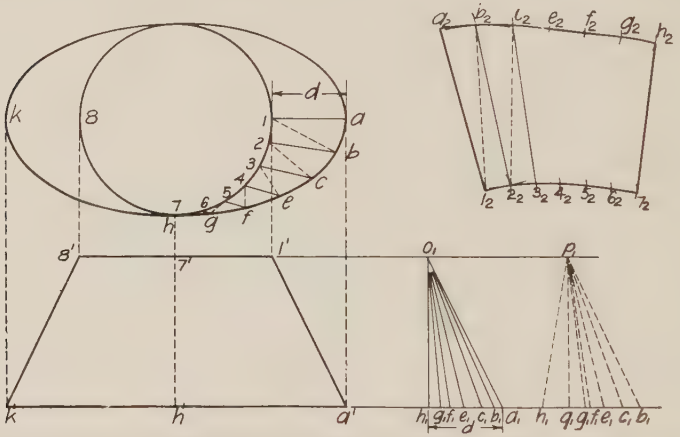


FIG. 92.—*Method for approximate development of warped surface.*

allel planes and they need not be of the form given in Fig. 92. If the surface is such that the straight line elements A1, B2, C3, etc., are neither parallel nor intersecting, it is warped and may be approximately developed by the following method.

Divide one quarter of the circle into any number of equal parts as six, Fig. 92. Divide the corresponding quarter of the ellipse into the same number of equal parts and join

points on the ellipse with the points on the circle by the lines A1, B2, C3, etc. Draw the dashed diagonal lines B1, C2, E3, etc. Find the true lengths of both of these sets of lines. A convenient method for getting these lengths is shown in the sketches to the right of the front view. The lengths of the top views of the lines are laid off on the base line from the foot of the perpendiculars from o_1 and p_1. The development is obtained by drawing the line a_2 1_2 in any direction and laying it off equal to the true length of A1. With 1_2 as center and the true length of B1 as radius strike an arc. With a_2 as center and ab as radius strike an arc cutting the first arc at b_2. Then with b_2 as center and B2 as radius and 1_2 as center and 12 as radius strike arcs intersecting at the point 2_2. By continuing this process the development of the entire piece is found. Only one-quarter of the development is shown, but the remainder is found in a similar manner.

175. **Approximate development of a sphere.** Theoretically, double curved surfaces can not be developed, but approximate developments are made and used in practice. There are two methods used for developing a sphere.

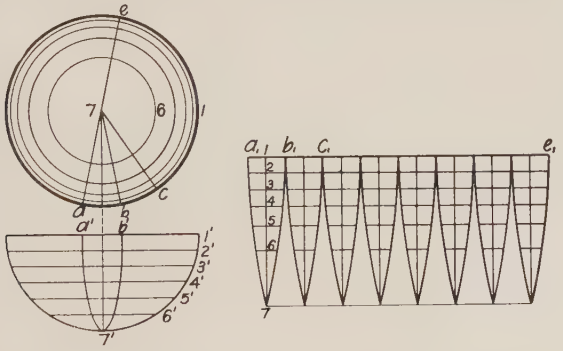


FIG. 93.—*Gore method for the development of a sphere.*

Method I. **The gore method.** Let the sphere be given as shown in Fig. 93. Only half the surface is shown and only one-quarter of the sphere is developed. but this will suffice to explain the method. Divide the surface into gores by the meridian planes A7, B7, C7, etc. Sixteen is

a good number although the more parts taken the more accurate the development should be. Then divide the surface by a series of horizontal circles 1, 2, 3, etc. In the development the great circle 1 becomes the straight line $a_1\ b_1$----e_1----, the distance a_1b_1=arc ab, $b_1\ c_1$=arc bc, etc. The line *17* in the development is at right angles to $a_1\ e_1$ at the mid point of $a_1\ b_1$, and is equal in length to one-quarter of a great circle, the distance *12*=arc *1'2'*, *23*=arc *2'3'* etc. Center lines for each gore are drawn from the proper points along the line a_1e_1 and these center lines divided into parts similar to the first one by lines through points *2, 3, 4,* etc., parallel to a_1e_1. Step off from the center line of each gore along the lines *2, 3, 4,* etc., distances obtained from the corresponding circles in the top view, measured from the center line to the sides of the gore. Joining the points thus located gives the development shown in

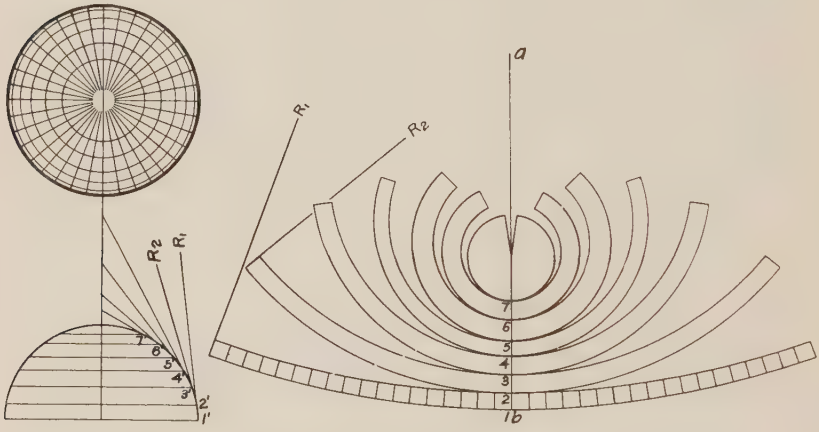


FIG. 94.—*Zone method for the development of a sphere.*

Fig. 93. The development of the complete sphere would consist of twice as many gores as shown, the half of each gore above the line $a_1\ e_1$ being similar to the half shown below this line.

Method II. **The zone method.** Let the sphere be given as in Fig. 94. Half the sphere is shown in the top and front views and the same half in the development.

Divide the surface of the sphere into zones by the horizontal planes 1, 2, 3, etc. The greater the number of zones the greater should be the accuracy of the development. Each zone is then developed as if it were the frustrum of a right cone. The radius of circle 1 in the development is the length of the line *1′ 2′* from the point *1′* to where *1′ 2′* cuts the vertical axis of the sphere. Two circles pass through the point 2 in the development, one with a radius from *2′* to the point where the line *1′ 2′* cuts the axis of the sphere, and the other with a radius from the point *2′* to where *2′ 3′* cuts this axis. By continuing in this manner, all the circles shown in the development are drawn. It is best to take a center line *ab* and have the centers of all these circles on this line. Meridian planes are then passed through the axis of the sphere dividing the surface into any convenient number of parts as shown in the top view. With dividers, step off the distances on circle 1 between these planes in the top view upon circle 1 of the development, beginning with the center line *ab* and stepping half of them off on each side of this line. Each zone is stepped off in this way from the top view upon the development. In Fig. 94 division lines are shown for the first zone only.

The developed zones for the other half of the sphere would have centers on the line *ab* extended and would be curved in the opposite direction. The other half would be an exact duplicate of the half shown. Patterns cut from such a development should give an approximate sphere when bent into proper position.

CHAPTER VII

INTERSECTIONS OF SURFACES

INTERSECTIONS OF THE SURFACES OF PRISMS AND PYRAMIDS

176. General method. Find where the edges of each solid pierces the faces of the other solid. A series of straight lines joining in order these piercing points is the required line of intersection.

The construction for such a problem can be made by either method of finding where a line pierces a plane. If the surfaces are both pyramids, the method of finding where a line pierces a plane without the aid of an auxiliary view will be found the most convenient. If, however, one of the surfaces is a prism, the best method for finding the intersection is to take an auxiliary view looking in the direction of the axis of the prism. This view will show the faces of the prism as lines and the intersections of these faces with the edges of the other surface are readily found. This method is particularly desirable when the axis of the prism is parallel to H or V. If the axis is oblique to both H and V, a second auxiliary view will be necessary to make the faces of the prism appear as lines. To find where the edges of the prism pierce the faces of the other solid, use the second method of finding where a line pierces a plane.

177. To find the line of intersection of the surfaces of a prism and a pyramid.

Let the prism and pyramid be given as in Fig. 95.

Analysis. The auxiliary view in the direction of the axis shows the prism as a square. From this view, the points in which the edges of the pyramid pierce the faces of the prism are found by inspection. To find the points in which an edge of the prism pierces the pyramid, use an auxiliary plane containing this edge and parallel to the base of the

pyramid. This plane cuts lines from the faces of the pyramid which intersect the edge of the prism in the required points.

Construction. In the auxiliary view, the points in which the lateral edges of the pyramid pierce the faces of the prism are seen to be x_1, y_1, w_1, etc. The top views x, y, w,

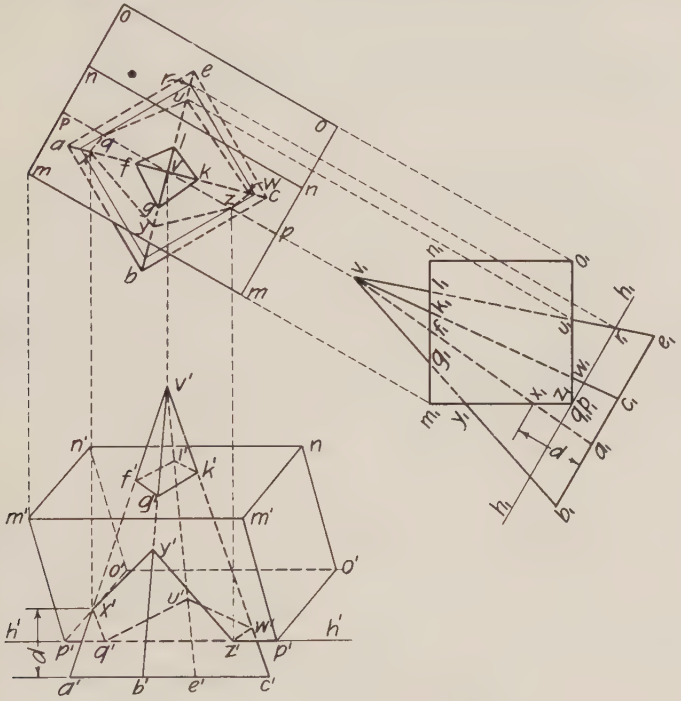


FIG. 95.—*Intersection of prism and pyramid.*

etc., of these points are found from the auxiliary view by projection and then the front views x', y', w', etc., are found from the top views. For the sake of accuracy, it is sometimes better to locate the front view before the top view. For example, locate x' on $v'a'$ by means of the distance d taken from the auxiliary view and then locate the top view x by projection. It is well to check the heights of the front views of the points of intersection above the base of the pyramid by means of distances such as d taken from the auxiliary view.

From the auxiliary view, it is seen that PP is the only edge of the prism which intersects the faces of the pyramid. To find where this edge pierces the face of the pyramid, use a horizontal plane containing this edge. $h'h'$ is the front and h_1h_1 the auxiliary view of this plane. The plane cuts a square from the pyramid which has one corner at R and sides parallel to the sides of the base of the pyramid. The square cuts the edge PP of the prism in the required points Q and Z. Joining the points thus found in the proper order gives the required line of intersection.

178. Problems.

Solve each of the following problems in a 11″ x 16″ rectangle with the base line through the center and parallel to the longer sides of the rectangle.

1. X (—4½, 2¼, —¼) Y (—4½, 2¼, —5¼) is the axis of a right square prism (side of base 2¼″). One face of the prism makes 15° with V. Find the line of intersection of this prism with a 2½″ cube. The diagonal of the cube coincides with the axis of the prism and the top view of one edge of the cube is perpendicular to a face of the prism. Develop the surface of the cube showing the line of intersection on the development.

2. The same as problem 1 except that the prism has an equilateral triangle with side 2½″ for a base.

3. The same as problem 1 except that the diagonal of the cube is ½″ from the axis of the prism.

4. Find the line of intersection of a square prism and a right square pyramid. Prism 2″ x 2″ x 4½″, with edges parallel to H and 30° with V. The plane of one face makes 30° with H and the lowest edge is ¾″ above the base of the pyramid. The base of the pyramid is a 2½″ square resting on a horizontal plane, with center at O (—4, 2½, —5¼), side of base 60° with V. The axis of the pyramid is perpendicular to H and is 5″ long. The axes of the prism and the pyramid are ⅛″ apart.

NOTE.—This problem can be multiplied by changing the number of faces of the prism or pyramid, by changing the distance between the axes, by changing the angle which the plane of the face of the prism makes with H, or by raising or lowering the prism.

INTERSECTIONS OF CURVED SURFACES

179. To find the line of intersection of any two surfaces. In general, pass a system of planes through the surfaces. The planes will cut lines from each surface and the intersections of these lines will be points on the required line of intersection. The planes should be passed so that they will cut the simplest lines possible from the given surfaces.

Sometimes the intersection can be more easily found by using a system of spheres in place of the system of planes. For example, if the axes of two surfaces of revolution intersect, a system of spheres having their centers at the intersection of the axes and radii of different lengths, will cut circles from each of the original surfaces. The intersections of these circles are points on the required line of intersection.

When one of the surfaces is a cylinder, particularly a right circular cylinder, an auxiliary view taken by looking in the direction of the axis of the cylinder will be found a convenient method for getting the intersection.

180. To find the line of intersection of two right circular cylinders.

Let the cylinders be given as in Fig. 96.

Analysis. Pass a system of planes parallel to the axes of both cylinders. These planes will cut straight line elements from each cylinder which will intersect in points of the required curve of intersection.

Construction. The planes which are parallel to the axes of both cylinders appear as straight lines in the front and also in the auxiliary view. $t'\ t'$ represents one such plane in the front view. This plane is located in the auxiliary view by taking the distance d which $t'\ t'$ is above $m'\ n'$ in the front view and setting it above $h_1\ h_1$, thus locating $t_1\ t_1$ which is the auxiliary view of the plane. The auxiliary view $t_1\ t_1$ might have been drawn first and then the front view $t'\ t'$ determined. This plane cuts two elements from each of the cylinders, and these elements intersect locating the points X, Y, Z, and W, on the curve of intersection of

the cylinders. By using other planes higher and lower than T, enough points can be located to determine the curve of intersection as shown in Fig. 96.

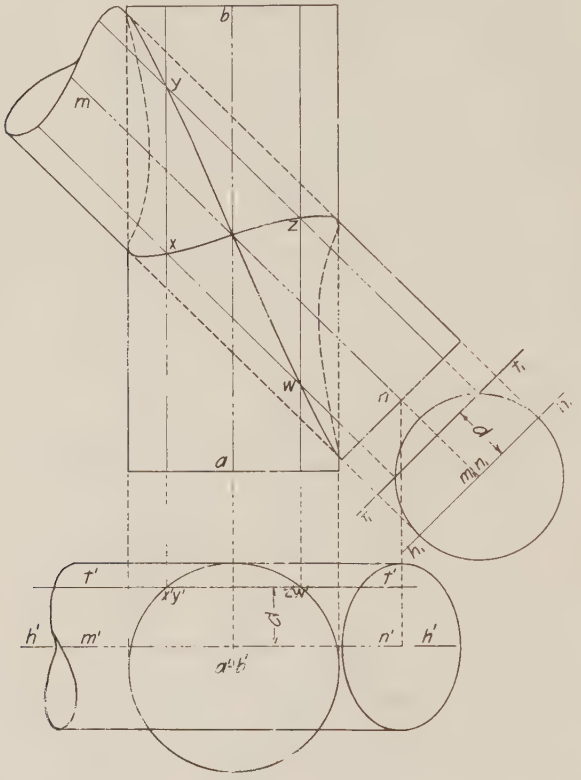


FIG. 96.—*Intersection of two right circular cylinders.*

To determine which part of the curve of intersection of two surfaces is visible. A point on the curve of intersection is visible in the top view if it lies on an element of each cylinder which is visible in that view. In like manner, a point is visible in the front view when it lies on an element of each cylinder which is visible in that view. This method is general and applies to the intersection of any two surfaces. It is a convenient method for determining which part of the curve of intersection of any two surfaces is visible or invisible in any view.

181. To find the line of intersection of a right circular cone and a right circular cylinder.

Let the cone and cylinder be given as in Fig. 97.

Analysis. 1st method. Pass a system of planes through the vertex of the cone and parallel with the axis of the cyl-

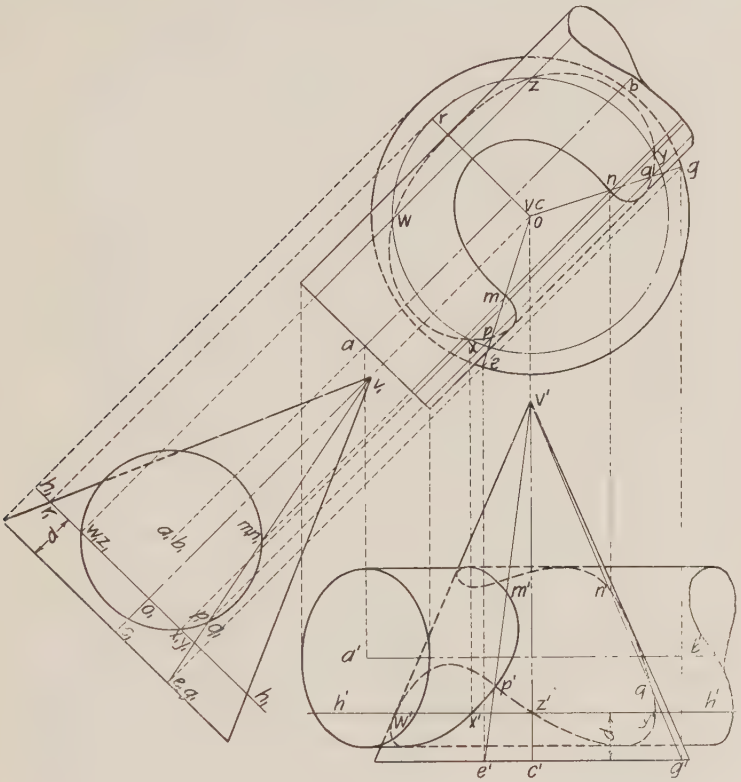


FIG. 97.—*Intersection of a right cylinder with a right cone.*

inder. These planes will cut straight line elements from each surface and these lines will intersect in points of the required curve of intersection.

2nd method. Pass a system of planes perpendicular to the axis of the cone. These planes will cut circles from the cone which will intersect the straight lines cut from the cylinder in points of the required curve of intersection.

Construction. 1st method. Take an auxiliary view by looking in the direction of the axis of the cylinder. In this view v_1 e_1 represents a plane which passes through the vertex of the cone and is parallel to the axis of the cylinder. This plane cuts the elements MN and PQ from the cylinder and the elements VE and VG from the cone. These elements intersect in the points M, N, P, and Q of the required curve of intersection. The front views of points on the curve of intersection, such as M and P, can usually be located accurately by drawing the front views of the elements of the cylinder and projecting to those elements from the top views of the points. In a similar manner other points can be located and the curve of intersection as shown in Fig. 97 determined.

2nd method. If the planes are passed according to the second method given in the analysis, they will appear as lines parallel to the base of the cone in the auxiliary view. h_1 h_1 is the auxiliary view of one of these planes. It cuts the elements XY and WZ from the cylinder and a circle of radius o_1 r_1 from the cone. The circle cuts the elements of the cylinder at the points X, Y, Z, and W of the required curve of intersection. Continuing in this manner, the curve of intersection as shown in Fig. 97 is determined.

When the surfaces are located as in Fig. 97, the second method gives a more accurate construction than the first, for most of the points on the curve of intersection. However, points on the element VR of the cone and elements near it can be more accurately located by the first method. The student should be able to use both methods and should use his judgment as to which gives the more accurate construction.

182. To find the line of intersection of two oblique cylinders.

Let the cylinders be given as in Fig. 98.

Analysis. Pass a system of planes parallel to the axes of both cylinders. These planes will cut straight line elements from each cylinder which will intersect in points of the required curve of intersection.

Construction. Through any point A draw two lines, one parallel to the elements of one cylinder and the other parallel to the elements of the other cylinder. The plane of these lines cuts the plane of the bases of the cylinders in the line *tt*. Since the planes are parallel, each of them will cut a line from the plane of the bases which is parallel

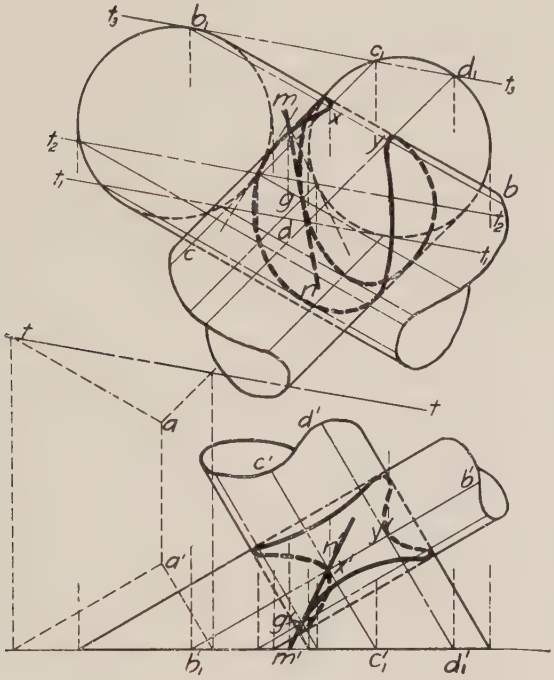


FIG. 98.—*Intersection of two oblique cylinders.*

to *tt*. The plane T_3 which is tangent to one of the cylinders, cuts the element BB_1 from that cylinder and the elements CC_1 and DD_1 from the other cylinder. These elements intersect in the points X and Y, points of the required line of intersection. The plane T_1 is tangent to the cylinder on the right and cuts two elements from the cylinder on the left. The intersections of these elements determine two other points on the curve of intersection. Any plane, as T_2, parallel to and between planes T_1 and T_3 will

cut two elements from each cylinder, thus locating four points on the curve of intersection. A curved line joining in the proper order the points thus found is the required line of intersection of the cylinders.

For construction purposes, it is much better to have a few well chosen planes than many poorly chosen ones. Planes such as T_1 and T_3, tangent to one surface and cutting the other, should always be drawn. All other planes should be drawn between these as the surfaces do not cut each other outside of these planes. Planes containing the outside or bounding elements of the cylinders in both top and front views locate the points where the curve of intersection is tangent to these elements. These being points of tangency are more helpful in drawing the curve of intersection than ordinary points of the curve would be. These tangent points are also the points in which the curve changes from visible to invisible if it changes at all.

To draw a tangent to the curve of intersection at any point. Select the point at which the tangent is to be drawn and pass a plane tangent to each cylinder at this point. The intersection of these planes is the required tangent to the curve. Each tangent plane is determined by the element through the point of contact and a tangent to the curve of the base at the point in which the element cuts the base. In Fig. 98, the elements of the cylinders intersect at G and the tangents to the base circles intersect at M. Then MG is the required tangent to the curve of intersection of the cylinders.

183. In the preceding problem, the bases of both cylinders are taken in one plane. If the bases are not both in one plane, the elements of either cylinder can be extended until they cut the plane of the base of the other cylinder, thus locating a new base. The problem can then be solved as before. Another method of solution in case the bases of both cylinders do not lie in one plane is to find the lines of intersection of the cutting planes with the planes of the bases of both cylinders. From the points where these lines cross the curves of the bases, draw elements of the cylin-

ders which will intersect in points of the required curve of intersection.

If the bases of both cylinders are circles in the same plane, as shown in Fig. 98, or if the bases are circles in parallel planes, points on the curve of intersection can be found by passing planes parallel to the plane of the bases. Each of these planes will cut circles from the cylinders and these circles will intersect in points of the required curve of intersection.

In case a model of the cylinders as shown in Fig. 98 is desired, it is necessary to develop both surfaces by the method of Art. 166. The line of intersection of the surfaces should be laid off on the development of one cylinder, and if a free passage through both cylinders is desired, it must be laid off on the development of each cylinder. With the developments as patterns, material can be cut which will roll into the desired form.

184. To find the line of intersection of a cone and cylinder.

Let the surfaces be given as in Fig. 99.

Analysis. Pass a system of planes through the vertex of the cone and parallel with the elements of the cylinder. These planes will cut from each surface straight line elements which will intersect in points of the required line of intersection. A line through the vertex of the cone and parallel with the elements of the cylinder will lie in all of the planes.

Construction. The line V_1X, through the vertex of the cone and parallel with the elements of the cylinder, lies in all the planes and pierces the plane of the bases at X, a point common to all of the lines which these planes cut from the plane of the bases. x' is first determined and then x. Through x, draw the lines xt, xt_1, etc., cutting the bases of both surfaces. From the points where these lines cross the bases draw elements of the surfaces. These elements will intersect in points of the required line of intersection. In this manner the curve of intersection as shown in Fig. 99 is found.

The visible part of the curve is determined and the tangent is drawn in the same manner as in Arts. 180 and 182. If the bases of the surfaces are not in the same plane, the methods suggested in Art. 183 will solve the problem.

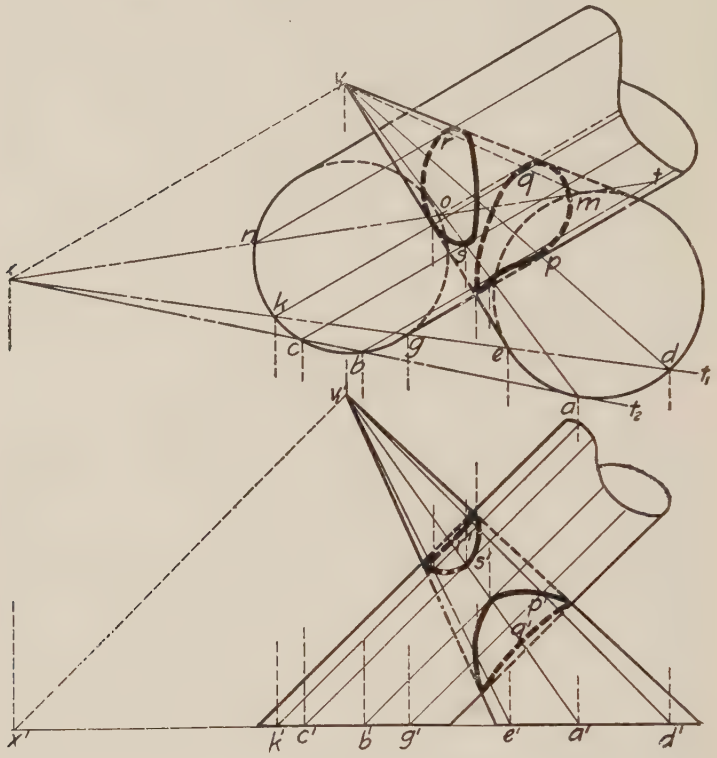


FIG. 99.—*Intersection of an oblique cone with an oblique cylinder.*

As with the cylinders, the problem can be solved with a series of planes parallel with the plane of the bases, providing the bases of both surfaces are circles lying in one plane.

185. To determine in advance the nature of the curves in which cylinders and cones intersect. In the system of planes used in Fig. 99, there are two, T and T_2, which are tangent to the cone. xt and xt_2 are the intersections of these planes with the plane of the bases of the surfaces. The

plane T is tangent to the cone on the back side and the plane T_2 is tangent to the cone on the front side. Both of these tangent planes cut the cylinder. It is evident then that the cylinder extends beyond the cone on both sides; the cone running into the cylinder from one side and out on the other side, forming two distinct curves of intersection, as shown in the figure. If one of these tangent planes had cut the base of the cylinder while the other did not, there would have been one continuous curve of intersection. If the cylinder had been between these planes and not cut either of them, there would again have been two distinct curves of intersection, where the cylinder runs into the cone from one side and where it runs out on the other. In the case where the planes are tangent to both cone and cylinder, the two curves of intersection just touch each other.

The same method may be used to determine the nature of the curve of intersection of two cylinders as shown in Fig. 98, or of two cones as given in the next article.

186. To find the line of intersection of two cones.

Analysis. Pass a system of planes through the vertices of both cones. These planes will cut straight lines from each cone which will intersect in points of the required line of intersection. A line joining the vertices of the cones will lie in all of these planes and will pierce the plane of the bases in a point common to all the lines which these planes cut from the plane of the bases.

Let the construction be made for finding the curve of intersection. Also determine which part of the curve is visible and draw a tangent to the curve by methods previously given.

In what other way can the cutting planes be passed to find the curve of intersection?

187. To find the line of intersection of a sphere and a cone or a sphere and a cylinder.

Analysis 1. If the cone or cylinder has a circle for a base, the planes can be passed parallel to the base. Each plane will cut a circle from the sphere and a circle from

the cone or cylinder. The intersections of these circles are points on the required line of intersection.

Analysis 2. Take a series of planes which contain the rectilinear elements of the cone or cylinder and are perpendicular to H or V. These planes will cut circles from the sphere which will intersect the elements cut from the cone or cylinder in points of the required line of intersection. To find these points, it may be necessary to revolve the planes until they are parallel to one of the planes of projection. In the case of the cylinder, an auxiliary view which shows the circles cut from the sphere in their true sizes gives a satisfactory solution.

Let the constructions be made in accordance with the above analyses.

To draw a straight line tangent to the curve of intersection at any point. The required tangent will be the line of intersection of two planes, one tangent to the cone or cylinder and the other tangent to the sphere at the given point.

188. **To find the line of intersection of a torus and cylinder when their axes are perpendicular to each other.**

Analysis. Pass a system of planes which will cut both surfaces and which are perpendicular to the axis of the torus. Each of these planes will cut circles from the torus and straight lines from the cylinder. The intersections of these circles with the straight lines are points on the required line of intersection of the surfaces.

Construction. Take an auxiliary view of both surfaces by looking in the direction of the axis of the cylinder. In this view, the cutting planes will appear as straight lines. From the auxiliary view of the points where these lines cross the curves representing the surfaces, projecting lines are drawn to the view of the surfaces taken by looking in the direction of the axis of the torus. This view will show the points in which the circles cut from the torus intersect the lines cut from the cylinder. The line joining in proper order the points thus found is the required line of intersection of the torus and cylinder.

189. General method for finding the line of intersection of two surfaces A and B. Pass a system of surfaces, usually planes, in such a way that they will cut both of the given surfaces. Each of the third surfaces will cut a line from surface A which will intersect the line which it cuts from surface B in points of the line of intersection of A and B. The third surfaces should be passed in such a direction that they will cut the simplest lines possible from the surfaces A and B, and these lines should be in simple position with respect to the planes of projection. If possible cut either straight lines or circles from the given surfaces for these are the simplest lines to draw with a straight edge and compass.

GENERAL PROBLEMS—INTERSECTIONS OF SURFACES

Solve each of the following problems in a 11″ x 16″ rectangle with the base line through the center of the rectangle parallel to its longer side.

1. A (—5, 0, —1) B (—3, 0, —5) C (—1½, 0, —2) is the base of a prism whose edges are perpendicular to V. Find the line of intersection of this prism with a sphere (4½″ diameter) whose center is at O (—3, 2½, —3). Develop the surface of the prism showing the curve of intersection on the development.

2. Find the line of intersection of a sphere with a right circular cylinder. The center of the sphere is at C (—4, 2½, —2½), radius 2″. The axis of the cylinder is parallel to H and 30° with V, passing through the point A (—4, 2, —1½), radius of base 1″. The axis of the cylinder runs backward to the left. Develop one-fourth of the cylinder, showing the curve of intersection on the development.

3. Find the line of intersection of a right circular cone and a sphere. The center of the base of the cone is (0, 4¼, —3¾), radius of base 1¾″, altitude 5″. Center of sphere (—1⅛, 3½, —1¾), radius 1½″. Develop one-fourth of the cone, and show the curve of intersection on the development. (Take base line parallel to shorter side of sheet for this problem).

4. V (3½, 2½, —½) is the vertex of a right circular cone whose base is a circle (4½″ diameter) in H with center at C (3½, 2½, —5). Find the line of intersection of the cone with the prism M (½, 2, —2) N (1, 0, —4¼) O (0, 2¾, —4¾), X (6½, 4½, —2) Y (7, 2½, —4¼) Z (6, 5¼, —4¾).

5. V (4, 6, 0) is the vertex of an oblique cone whose base is a circle (4″ diameter) in H with center at O (—2, 2½, —7½). A (1¾, 3½, —½) B (1¾, 3½, —7½) is the axis of a right circular cylinder (3½″ diameter). Find the line of intersection of the cone with the cylinder. (G. L. parallel to shorter side of 11″ x 16″ rectangle.)

6. X (—1, 0, —3¼) Y (—6, 3¼, —3¼) is the axis of a right circular cone, base (3½″ diameter) with center at Y. A (—4¾, 2, —1) B (—4¾, 2, —5¼) is the axis of a right circular cylinder (3″ diameter). Find the line of intersection of the cone with the cylinder. Develop the surface of one-fourth of the cone or cylinder showing the curve of intersection on the development.

7. Find the line of intersection of a right circular cone with a right circular cylinder. The axis of the cone is A (—4, 1⅜, —4) B (—4, 1⅜, —½), radius of base 1½″. The axis of the cylinder is M (—6, 1½, —3) N (—1½, 1½, —3), radius of base 1″. Develop the surface of one-fourth of the cone or cylinder showing the curve of intersection on the development.

8. Find the intersection of a torus and a right circular cylinder. The axis of the torus is perpendicular to H, through the point (—1½, 3, —3½), radii of circles representing the surface in the top view ⅞″ and 3⅜″ respectively, centers of circles representing the torus in the front view 3½″ below the center line. The axis of the cylinder is A (—3¼, —1, —3¾) B (—3¼, 5, —3¾), radius of base 1⅜″. Develop one-fourth of the cylinder, showing the curve of intersection on the development. (Take base line parallel to the shorter side of sheet for this problem.)

9. Find the line of intersection of a cylinder and a sphere. The base of the cylinder is a circle in a horizontal plane, center at A (5, 2, —4½) and radius 1¾″. The axis makes 45° with H, is parallel with V, and slopes upward to the left. The center of the sphere is at O (3, 2⅜, —2¼), radius 2″. Develop one-quarter of the cylinder, showing the curve of intersection on the development.

10. Find the line of intersection of an oblique cylinder with a right circular cone. The bases of both surfaces are circles in the same horizontal plane. The center of the base of the cylinder is A (—¾, 2⅜, —4), radius 1⅜″, the elements are parallel to V, and 30° to H, sloping upward to the left. The center of the base of the cone is B (—4¼, 2¾, —4), radius 2″, altitude 4½″. Develop one-fourth of the cone, showing the curve of intersection on the development.

11. A (—6, 1¾, —1¾) B (—2½, 1¾, —1¾) is the axis of a right square prism. The plane of one face of the prism makes 30° with H, side of base 1½″. Find the intersection of this prism with a sphere with center at C (—4¼, 1½, —1½), radius 1″. Develop two faces of the prism, showing the curve of intersection on the development.

12. Find the line of intersection of a sphere and a right circular cylinder. The axis of the cylinder is A (—6, 3½, —3¼) B (—2½, 0, —3¼), radius of base ⅞″. The center of the sphere is O (—4½, 2¾, —2¾), radius 1½″. Develop one-fourth of the cylinder, showing the curve of intersection on the development.

13. Find the line of intersection of two cones which have their circular bases in the same horizontal plane. The center of the base of one cone is at A (1¼, 2½, —4½), vertex at O (—6½, 2½, ½), radius of base 1¾″. The center of the base of the other cone is at B (—2½, 2, —4½), vertex at C (—2½, 2, —1), radius of base 1¾″.

14. Find the intersection of a right circular cylinder with a right circular cone. The center of the base of the cone is at C (—4½, 3¼, —4½), vertex V (—4½, 3¼, 0), radius of base 2″. The center of the base of the cylinder is O (—2, 2, —3), axis parallel to H, 30° to V, sloping backward to the left; radius of base of cylinder 1¼″. Do not draw the front view of the bases of the cylinder unless they are used in getting the curve of intersection. Develop one-quarter of the cone or cylinder, showing the curve of intersection on the development.

15. Find the intersection of two right circular cylinders. The axes are A (—4, —1, —3¼) B (—4, 4¾, —3¼) and C (—1¾, —¼, —2⅞) E (—6, 4, —2⅞), and the diameters are 2½″ and 1¾″ respectively. Develop the surface of one cylinder showing the curve of intersection on the development.

16. Find the intersection of a cylinder and a right circular cone. The base of the cylinder is a horizontal circle with center at A (¾, 1⅝, —4¾), diameter 2¾". The axis of the cylinder is parallel to V, makes 30° with H, and slopes upward to the left. B (—2¾, 2, —¼) C (—2¾, 2, —4¾) is the axis of the cone whose base is a horizontal circle 4" in diameter with center at C.

17. Find the line of intersection of the two given oblique cylinders, which have their bases in a horizontal plane, and draw a tangent to the curve at some point of the curve not on an extreme element of either surface.

Cylinder A { Center of lower base, A (1⅛, X, —5¼)
Center of upper base, O (—3½, 3¾, —1¼)
Radius of base, 1¾"

Cylinder B { Center of lower base, B (—3½, Y, —5¼)
Center of upper base, C (½, 3¼, —1¼)
Radius of base, 1¾"

NOTE.—X may vary from 1¾" to 2¾" and Y may vary from 1¼" to 1⅞" as the instructor assigns.

18. Find the line of intersection of the given oblique cylinder and the given oblique cone, which have their bases in a horizontal plane, and draw a tangent to the curve at some point of the curve not on an extreme element of either surface.

Cylinder A { Center of the lower base, A (¼, X, —4¾)
Center of the upper base, C (—3, 1¾, —1¼)
Radius of the base, 1½"

Cone B { Center of the base, B (—3¼, Y, —4¾)
Vertex, O (1½, 4¼, 0)
Radius of the base, 1¾"

NOTE.—X may vary from 1½" to 2⅛" and Y may vary from 1⅛" to 1½" as the instructor assigns.

19. Find the line of intersection of two oblique cones which have their bases in a horizontal plane, and draw a line tangent to the curve of intersection at some point of the curve not on an extreme element of either surface.

Cone A { Center of the base, A (4¼, X, —4¾)
Vertex of the cone, O (1, 3¼, —1⅛)
Radius of the base, 2¼"

Cone B { Center of the base, B (0, Y, —4¾)
Vertex of the cone, C (6½, 4¼, 1½)
Radius of the base, 1¾"

NOTE.—X may vary from 2" to 2¾" and Y may vary from 2¼" to 2¾" as the instructor assigns.

20. Find the line of intersection of the two given cones, which have their bases in a horizontal plane, and draw a line tangent to the curve of intersection at some point of the curve not on an extreme element of either surface.

Cone A { Center of the base, A (0, 3½, —6¾)
Vertex of the cone, V (0, 3½, 0)
Radius of the base, 3″

Cone B { Center of the base, B (¼, 4, —6¾)
Vertex of the cone, C (—3¼, 3, —1¼)
Radius of the base, 2⅞″

NOTE.—Take G. L. through the center of the sheet and parallel to the shorter side.

21. A circle, 3½″ diameter, in H with center at C (—2½, 3¼, —4½) is the upper end of a vertical cylindrical pipe X which is connected by means of a breeches joint to two other vertical cylindrical pipes, Y and Z. The lower end of pipe Y is a horizontal circle, 1½″ diameter, with center at A (—½, 3¼, —2). The lower end of pipe Z is a horizontal circle, 1½″ diameter, with center at B (—4½, 3¼, + 1). Find the top and front views of the line of intersection of the pipes connecting the upper end of pipe X with the lower ends of pipes Y and Z. Develop one half of the pipe joining X and Y, or X and Z, as directed by the instructor.

22. A horizontal circle (3″ diameter) with center at M (—2⅛, 4, —6) is the section of a cylindrical smoke stack whose axis is M (—2⅛, 4, —6) N (—⅜, 5, —2). The opening in the deck of the ship around the lower end of the stack acts as a ventilator for the engine room. This opening is protected from the weather by a conical watershed, whose lower base is a horizontal circle (5″ diameter) with center at C (—1¾, 4, —6) and whose vertex is at V (—1¾, 4, —3½). (a) Find the top and front views of the line of intersection of the stack with the cone, showing hidden portions by dashed lines. (b) Construct the pattern for the right, front, one quarter of the conical watershed. (Take the base line parallel to the shorter sides of the rectangle).

23. A circle (4″ diameter) in V with center at C (—5½, 5, —3¼) is the front end of a horizontal cylindrical pipe X, which is connected by means of a breeches joint to two other horizontal cylindrical pipes Y and Z. The end of pipe Y is a circle (2″ diameter) with center at A (—5½, 1, —3¼). The end of pipe Z is a circle (1½″ diameter) with center at B (—2½, 2, —1½). (a) Find the top and front views of the line of intersection of the pipes connecting the pipe X with the pipes Y and Z. (b) Construct the development for the upper right one quarter of pipe connecting X with Y, showing the intersection on the development.

24. B (—4½, 2¼, —5¼) is the center of the upper end of a vertical circular cylindrical pipe W (4″ diameter). This pipe W is connected to 3 vertical circular cylindrical pipes X, Y, and Z by a 3 way sheet metal pipe fitting. Pipes X, Y, and Z are each 1¼″ in diameter. M (—1½, 2¼, —2), N, and O, the centers of the lower ends of pipes X, Y, and Z, are spaced 120° apart on a horizontal circle with center at A (—4½, 2¼, —2). (a) Find the top and front views of the line of intersection of any two of the three branches of the 3 way fitting. (b) Make a pattern for one-half the surface of one branch of the 3 way fitting, showing upper base, lower base, and curve of intersection with adjacent branch. Start pattern with longest or shortest edge.

DEFINITIONS AND CONSTRUCTIONS FROM PLANE AND SOLID GEOMETRY

A tetrahedron is a solid bounded by four plane faces. It is a pyramid with a triangular base.

A prism is a solid bounded by plane faces, two of which (the bases) are equal polygons in parallel planes and the other faces are parallelograms. The prism has three or more faces. The **lateral edges** of the prism are the intersections of the **lateral faces. An oblique prism** is one whose lateral edges are oblique to the plane of the base. **A right prism** is one whose lateral edges are perpendicular to the plane of the base.

A pyramid is a solid bounded by plane faces, one of which, called the **base,** is a polygon of any number of sides and the other faces are triangles having a common vertex. The **lateral edges** of the pyramid run from the corners of the base to the vertex. A pyramid is **regular** if its base is a regular polygon whose center coincides with the foot of the perpendicular let fall from the vertex to the plane of the base.

A parallelopiped is a solid all of whose faces are parallelograms.

GEOMETRICAL CONSTRUCTIONS

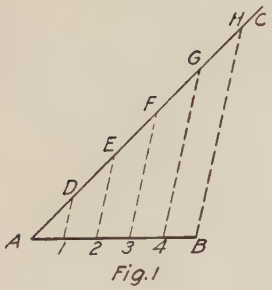


Fig. 1

1. To divide a given straight line into any number of equal parts. (Fig. 1).

Let it be required to divide the given straight line AB into five equal parts.

From either end of the line, as A, draw a line AC of indefinite length and making any angle with AB. Beginning with A, lay off on AC five equal parts of any convenient length as AD, DE. . . GH. This is easily accomplished with the dividers. Join H with B and then through the points D, E, F, and G draw lines parallel to HB. These lines cut AB in the points 1, 2, 3, and 4, and these points divide AB into the required number of equal parts.

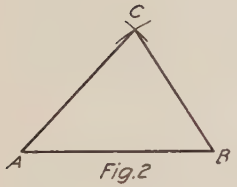


Fig. 2

2. To construct a triangle having given the lengths of its three sides. (Fig. 2).

Let the sides be 1½", 1¾", and 2" long, respectively.

Draw a straight line AB, 2" long. With A as center and a 1¾" radius strike an arc. With B as center and a 1½" radius, strike an arc cutting the first arc at C. Join C with both A and B by straight lines. Then ABC is the required triangle.

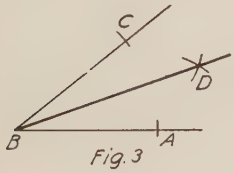


Fig. 3

3. To bisect an angle. (Fig. 3).

Let ABC be the given angle.

With B as center and any radius, draw an arc cutting the sides of the angle at A, and C. With A and C respectively as centers and any radius draw arcs intersecting at D. Draw the straight line BD; this is the required bisector.

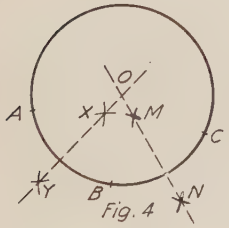


Fig. 4

4. To draw a circle through three given points not in the same straight line. (Fig. 4).

Let A, B, and C be the given points.

With A and B respectively as centers, draw arcs of equal radii intersecting at X and Y. With B and C respectively as centers, draw arcs intersecting at M and N. The line joining X and Y intersects the line joining M and N at O, the center of the required circle.

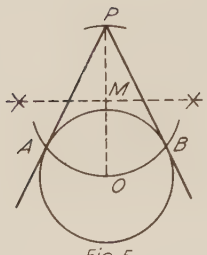


Fig. 5

5. To draw tangents to a circle from a given point without the circle. (Fig. 5).

Let O be the center of the given circle, and P be the given point.

Bisect the line PO, finding its midpoint M. With M as center and MO as radius, draw arcs intersecting the given circle at A and B. Then PA and PB are the required tangents. A and B are the points of tangency.

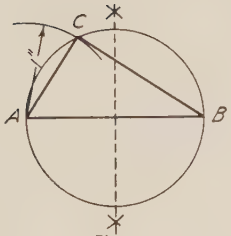


Fig. 6

6. To construct right triangles having a given hypotenuse. (Fig. 6).

Let AB be the hypotenuse.

There are any number of right triangles which can be drawn with AB as the hypotenuse. All of these triangles, however, can be inscribed in the circle having AB as a diameter.

Let it be further required that one side be 1″ in length. Draw the circle having AB as its diameter. With one end of the diameter (A) as center, strike an arc with 1″ radius cutting the circle at C. Then CAB is the required right triangle.

The problem could have been solved with the angles at A or B of any required size less than 90°.

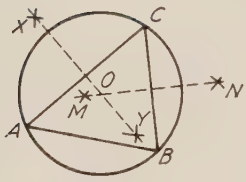


Fig. 7

7. To circumscribe a circle about a given triangle. (Fig. 7).

Let ABC be the given triangle.

Draw the perpendicular bisectors, XY and MN, of any two sides of the triangle, as AC and BC. XY and MN intersect at O, the center of the required circle. The perpendicular bisector of the side AB would also pass through O.

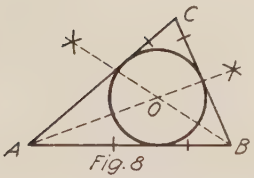


Fig. 8

8. To inscribe a circle in a given triangle. (Fig. 8).

Let ABC be the given triangle.

Bisect any two of the angles of the triangle as A and B. These bisectors intersect in the center O of the inscribed circle. The radius of the circle is the perpendicular distance from O to one of the sides of the triangle. The bisector of the angle C would also pass through the center O.

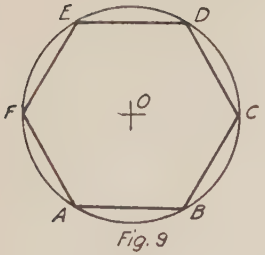


Fig. 9

9. To inscribe a hexagon in a given circle. (Fig. 9).

Let O be the center and OA the radius of the given circle.

The length of the side of a hexagon inscribed in a circle is equal to the radius of the circle. With O as center and OA as radius, draw the circle. Beginning with any point of the circumference, as A, and with a radius OA, locate B. Then with B as center locate C, etc. Joining the points thus located by straight lines forms the required hexagon ABCDEF.

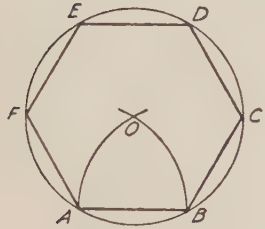


Fig. 10

10. To construct a hexagon upon a given straight line. (Fig. 10).

Let AB be the given straight line. With A as center and AB as radius, describe an arc. With B as center and the same radius, describe an arc cutting the first arc at O. With O as center and radius OA, draw a circle. This circle passes through B. Inscribe a hexagon in this circle by the method of Art. 9. This will be the required hexagon ABCDEF.

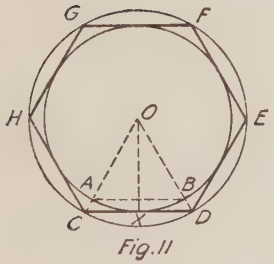


Fig. 11

11. To circumscribe a hexagon about a given circle. (Fig. 11).

Let O be the center, and OA the radius of the given circle.

Draw the given circle and then with A as center and AO as radius, locate the point B on the circumference. Bisect the angle AOB, thus locating X on the circumference. Erect a perpendicular to OX at X and produce it until it cuts OA and OB extended at C and D respectively. With O as center and OC as radius, draw a circle. This circle passes through D. With CD as a side, inscribe in the new circle the required hexagon CDEFGH by the method of Art. 9.